TRACES AND QUASI-TRACES ON THE BOUTET DE MONVEL ALGEBRA

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Abstract. We construct an analogue of Kontsevich and Vishik's canonical trace for pseudo-differential boundary value problems in the Boutet de Monvel calculus on compact manifolds with boundary. For an operator A in the calculus (of class zero), and an auxiliary operator B, formed of the Dirichlet realization of a strongly elliptic second-order differential operator and an elliptic operator on the boundary, we consider the coefficient $C_0(A,B)$ of $(-\lambda)^{-N}$ in the asymptotic expansion of the resolvent trace $\operatorname{Tr}(A(B-\lambda)^{-N})$ (with N large) in powers and log-powers of λ. This coefficient identifies with the zero-power coefficient in the Laurent series for the zeta function $\operatorname{Tr}(AB^{-s})$ at $s=0$, when B is invertible. We show that $C_0(A,B)$ is in general a quasi-trace, in the sense that it vanishes on commutators $[A,A']$ modulo local terms, and has a specific value independent of the auxiliary operator, modulo local terms. The local "errors" vanish when A is a singular Green operator of noninteger order, or of integer order with a certain parity; then $C_0(A,B)$ is a trace of A. They do not in general vanish when the interior ps.d.o. part of A is nontrivial.
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1. Introduction

In their work on the geometry of determinants of elliptic operators, Kontsevich and Vishik [KV] introduced the *canonical trace*, a novel functional $\operatorname{TR} A$ defined for a class of pseudodifferential operators (ψdo's) A on a closed n-dimensional manifold X. What makes it remarkable is that it extends the standard operator trace, and hereby complements the noncommutative residue.

The noncommutative residue $\operatorname{res} A$ was discovered by Wodzicki [W] and, independently, by Guillemin [Gu]; it is a trace (i.e., a nontrivial linear functional which vanishes on commutators) on the full algebra $\mathcal{A}$ of all classical pseudodifferential operators on X. The noncommutative residue turns out to be the only trace on $\mathcal{A}$ with this property, up to multiples. Moreover, the value of $\operatorname{res} A$ is determined from finitely many terms in the asymptotic expansion of the symbol of A (in fact, from the component of homogeneity $-n$ in this expansion); we call such functionals "locally determined".

It vanishes on operators of noninteger order or of order $< -n$, and it is here that Kontsevich and Vishik's functional takes over: $\operatorname{TR} A$ is well-defined and nontrivial when A is of noninteger order or of order $< -n$, and it vanishes on commutators $[A,A']$ of these types. It is "global" in that the value of $\operatorname{TR} A$ depends on the full operator and cannot be determined from the terms in the asymptotic expansion of the symbol. The canonical trace

equals the standard trace when the order is $< -n$; it is moreover defined on integer-order operators with certain parity properties, see below.

In this article we shall generalize the canonical trace to pseudodifferential boundary value problems in the calculus of Boutet de Monvel [B].

The original approach of [KV] is based on studies of generalized zeta functions $\zeta(A,P,s) = \operatorname{Tr}(AP^{-s})$ and certain regularizations of them, where P is an auxiliary elliptic operator. This does not extend easily to the situation of manifolds with boundary; in the case without boundary, P^{-s} is a classical ψdo again (by Seeley [S]), whereas complex powers of elliptic boundary problems fall outside the calculus of Boutet de Monvel. However, the introduction of $\operatorname{TR} A$ can instead be based on trace expansions of heat operators Ae^{-tP} (Lesch [L]) or resolvents $A(P-\lambda)^{-N}$ (Grubb [G4]), where the latter admit a direct generalization to manifolds with boundary (Grubb and Schrohe [GSc]).

Therefore, let us explain the functional of [KV] from the resolvent point of view: Let A be a classical ψdo of order ν; its symbol has an expansion in local coordinates $a \sim \sum_{j\ge 0} a_{\nu-j}$ with $a_{\nu-j}(x,\xi)$ smooth in (x,ξ) and homogeneous of degree $\nu - j$ in ξ for $|\xi| \ge 1$. When P is an auxiliary elliptic operator of integer order $m > 0$ with no principal symbol eigenvalues on $\mathbb{R}_-$, a calculation in local coordinates shows that the operator family $A(P-\lambda)^{-N}$ has for $N > (n+\nu)/m$ a trace expansion

$$\operatorname{Tr}(A(P-\lambda)^{-N}) \sim \sum_{j\ge 0} \tilde c_j (-\lambda)^{\frac{n+\nu-j}{m}-N} + \sum_{k\ge 0} \big(\tilde c_k' \log(-\lambda) + \tilde c_k''\big)(-\lambda)^{-k-N}, \tag{1.1}$$

for $\lambda \to \infty$ in a small sector around $\mathbb{R}_-$, cf. Grubb and Seeley [GS1]. Here each $\tilde c_j$ (and each $\tilde c_k'$) comes from a specific homogeneous term in the symbol of $A(P-\lambda)^{-N}$, whereas the $\tilde c_k''$ depend on the full symbol. So the coefficients $\tilde c_j$ and $\tilde c_k'$ depend each on a finite set of homogeneous terms in the symbols of A and P; we call such coefficients 'locally determined' (or 'local'), while the $\tilde c_k''$ are called 'global'. When $\nu \notin \mathbb{Z}$, the $\tilde c_k'$ vanish. When $\nu \in \mathbb{Z}$ and $(j-n-\nu)/m$ is an integer $k \ge 0$, both $\tilde c_j$ and $\tilde c_k''$ contribute to the power $(-\lambda)^{-k-N}$; their sum is independent of the choice of local coordinates, whereas the splitting in $\tilde c_j$ and $\tilde c_k''$ depends in a well-defined way on the symbol structure in local coordinates (see [GS1, Th. 2.1] or [G4, Th. 1.3]).

In (1.1), $m \cdot \tilde c_0' = \operatorname{res} A$, depending solely on A, cf. [W], [Gu]. Moreover, cf. [KV], [G4],

$$\operatorname{TR} A = \tilde c_0'' \tag{1.2}$$

in the following four cases:

(1) $\nu < -n$,
(2) $\nu \notin \mathbb{Z}$,
(3) $\nu \in \mathbb{Z}$, A is even-even and n is odd,
(4) $\nu \in \mathbb{Z}$, A is even-odd and n is even;

in the cases (3) and (4), P is taken to be even-even with m even. In all the cases (1)–(4), $\operatorname{res} A = 0$, there is no contribution to $(-\lambda)^{-N}$ from the first sum in (1.1), and

$$\tilde c_0'' = \int ⨍ \operatorname{tr} a(x,\xi)\, đ\xi dx, \tag{1.3}$$

where $⨍ \operatorname{tr} a(x,\xi)\, đ\xi$ is a finite part integral (explained in detail in Section 3 below).

When $\nu \in \mathbb{Z}$, we say that A or a has *even-even* alternating parity (in short: is even-even), when the symbols with even (resp. odd) degree $\nu - j$ are even (resp. odd) in ξ:

$$a_{\nu-j}(x,-\xi) = (-1)^{\nu-j} a_{\nu-j}(x,\xi) \text{ for } |\xi| \geq 1, \tag{1.4}$$

and the derivatives in x and ξ likewise have this property. A and a are said to have *even-odd* alternating parity in the reversed situation, where the symbols with even (resp. odd) degree $\nu - j$ are odd (resp. even) in ξ:

$$a_{\nu-j}(x,-\xi) = (-1)^{\nu-j-1} a_{\nu-j}(x,\xi) \text{ for } |\xi| \geq 1, \tag{1.5}$$

etc. These properties are preserved under coordinate changes. For brevity, we shall say that A (or a) *has a parity that fits with the dimension* n, when (3) or (4) holds.

[KV] only considered the cases (1)–(3); the operators satisfying (3) were called odd-class operators. (4) was included in [G4].

The relations can also be formulated in terms of the generalized zeta function $\zeta(A,P,s) = \operatorname{Tr}(AP^{-s})$; here we assume P invertible for simplicity. It is known from [W], [Gu] that $\zeta(A,P,s)$ extends meromorphically from large $\operatorname{Re} s$ across $\operatorname{Re} s = 0$ with a simple pole at 0,

$$\zeta(A,P,s) = C_{-1}(A,P)s^{-1} + C_0(A,P) + O(s) \text{ for } s \to 0; \tag{1.6}$$

this follows also by translating (1.1) to a statement on the pole structure of $\zeta(A,P,s)$ (as e.g. in [GS2, Cor. 2.10]). Here

$$\begin{aligned} C_{-1}(A,P) &= \tilde{c}'_0 = \tfrac{1}{m} \operatorname{res} A, \\ C_0(A,P) &= \tilde{c}_{\nu+n} + \tilde{c}''_0, \end{aligned} \tag{1.7}$$

with $\tilde{c}_{\nu+n}$ defined as 0 if $\nu < -n$ or $\nu \notin \mathbb{Z}$. In the cases (1)–(4), $\tilde{c}_{\nu+n}$ and $\operatorname{res} A$ vanish (for any choice of local coordinates), so that

$$\operatorname{TR} A = \tilde{c}''_0 = C_0(A,P) = \zeta(A,P,0). \tag{1.8}$$

The coefficient $C_0(A,P)$ has an interest also when none of the conditions (1)–(4) is satisfied. Then it has the properties:

$$C_0(A,P) - C_0(A,P') \text{ and } C_0([A,A'],P) \text{ are locally determined.} \tag{1.9}$$

Moreover, $C_0(A,P)$ can be written as (1.3) plus local terms. Functionals with properties as in (1.9) for a system of auxiliary elliptic operators P will be called quasi-traces.

By use of more functional calculus, defining $\log P$, one can moreover show that the two expressions in (1.9) are noncommutative residues of suitable combinations of the given operators and $\log P$; cf. [KV] and Okikiolu [O] for $C_0(A,P) - C_0(A,P')$, cf. Melrose and Nistor [MN] for $C_0([A,A'],P)$. (In works of Melrose et al., notation such as $\widehat{\operatorname{Tr}}(A)$ or $\operatorname{Tr}_P(A)$ is used for $C_0(A,P)$; it is called a regularized trace there.)

In order to extend the definition of the canonical trace to operators in the calculus of Boutet de Monvel [B] on an n-dimensional compact C^∞-manifold X with boundary

$\partial X = X'$, we shall rely on resolvent expansions analogous to (1.1). To this end we choose an auxiliary operator $P_{1,\mathrm{D}}$, which is the (invertible) Dirichlet realization of a strongly elliptic principally scalar second-order differential operator.

It was shown in [GSc] that when $A = P_+ + G$ is a pseudodifferential boundary operator (ψdbo) in this calculus of order $\nu \in \mathbb{Z}$ with G of class 0, then there is a trace expansion for $N > (n+\nu)/2$:

$$\operatorname{Tr}(A(P_{1,\mathrm{D}} - \lambda)^{-N}) \sim \sum_{j\ge 0} \tilde{c}_j (-\lambda)^{\frac{n+\nu-j}{2}-N} + \sum_{k\ge 0} (\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k)(-\lambda)^{-\frac{k}{2}-N}. \tag{1.10}$$

The proof depends on a reduction to parameter-dependent ψdo's on X', where [GS1] can be used; it is easily generalized to cases where $A = G$ of noninteger order ν and class 0. When $\nu \notin \mathbb{Z}$, the $\tilde{c}'_k$ vanish.

As explained e.g. in [GS2] this yields the expansions:

$$\operatorname{Tr}(Ae^{-tP_{1,\mathrm{D}}}) \sim \sum_{j\ge 0} c_j t^{\frac{j-n-\nu}{2}} + \sum_{k\ge 0} (-c'_k \log t + c''_k) t^{\frac{k}{2}}, \tag{1.11}$$

$$\Gamma(s) \operatorname{Tr}(AP_{1,\mathrm{D}}^{-s}) \sim \sum_{j\ge 0} \frac{c_j}{s + \frac{j-n-\nu}{2}} + \sum_{k\ge 0} \Big(\frac{c'_k}{(s+\frac{k}{2})^2} + \frac{c''_k}{s+\frac{k}{2}} \Big). \tag{1.12}$$

In (1.10), $\lambda \to \infty$ in a sector of $\mathbb{C}$; in (1.11), $t \to 0+$; (1.12) describes the pole structure of the meromorphic extension of $\Gamma(s)\operatorname{Tr}(AP_{1,\mathrm{D}}^{-s})$ to $\mathbb{C}$. The coefficients c_j, c'_k, c''_k are proportional to the coefficients $\tilde{c}_j, \tilde{c}'_k, \tilde{c}''_k$ in (1.10) by universal constants; in particular,

$$c'_0 = \tilde{c}'_0, \quad c''_0 = \tilde{c}''_0, \quad c_{\nu+n} = \tilde{c}_{\nu+n}, \tag{1.13}$$

where $c_{\nu+n}$ and $\tilde{c}_{\nu+n}$ are taken as 0 when $\nu < -n$ or $\nu \notin \mathbb{Z}$. The coefficients c_j and c'_k are locally determined, whereas the c''_k are global.

Relation (1.12) implies that the generalized zeta function $\zeta(A, P_{1,\mathrm{D}}, s) = \operatorname{Tr}(AP_{1,\mathrm{D}}^{-s})$ extends meromorphically with

$$\begin{gathered} \zeta(A, P_{1,\mathrm{D}}, s) = C_{-1}(A, P_{1,\mathrm{D}}) s^{-1} + C_0(A, P_{1,\mathrm{D}}) + O(s) \text{ for } s \to 0, \\ C_{-1}(A, P_{1,\mathrm{D}}) = \tilde{c}'_0, \quad C_0(A, P_{1,\mathrm{D}}) = \tilde{c}_{\nu+n} + \tilde{c}''_0. \end{gathered} \tag{1.14}$$

In [FGLS], an analogue of Wodzicki's noncommutative residue was introduced by Fedosov, Golse, Leichtnam, and Schrohe for operators in Boutet de Monvel's calculus. In [GSc] we were able to show that the coefficient $\tilde{c}'_0 = C_{-1}(A, P_{1,\mathrm{D}})$ in the above expansions satisfies the relation

$$\tilde{c}'_0 = \tfrac{1}{2} \operatorname{res} A. \tag{1.15}$$

We shall presently investigate $C_0(A, P_{1,\mathrm{D}})$ as a candidate for a canonical trace. We establish quasi-trace properties as in (1.9), with formulas for the "value modulo local terms". Moreover, we extract some cases where $C_0(A, P_{1,\mathrm{D}})$ is independent of $P_{1,\mathrm{D}}$ and vanishes on commutators $[A, A']$, so that it is a trace. Here we show in particular that $C_0([A, A'], P_{1,\mathrm{D}}) = 0$ whenever A and A' are singular Green operators of class zero and

orders ν and ν' with $\nu+\nu' < 1-n$ or $\nu+\nu' \notin \mathbb{Z}$; the same is true when $\nu+\nu'$ is an integer and a certain parity holds, provided we narrow down slightly the class of admissible auxiliary operators P_1. When $A = P_+ + G$ and $A' = P'_+ + G'$ both have nontrivial pseudodifferential part (and hence the orders are integer), however, one cannot hope for more than the quasi-trace property, see Remark 4.2 for a discussion of this point.

In contrast with the boundaryless case, the powers $P^s_{1,\mathrm{D}}$ are far from belonging to the ψdbo calculus when $s \notin \mathbb{Z}$, so we do not expect to find residue formulas as in [KV], [O], [MN].

The quasi-trace property has an interest in itself, for situations where one has some control over the local terms. Our formulas for $C_0(P_+, P_{1,\mathrm{D}})$ in Theorem 4.1 and for $C_0(G, P_{1,\mathrm{D}})$ in Theorem 3.6 moreover show how the values are related to formulas for the boundaryless manifolds $\widetilde{X}$ resp. X'.

2. Preliminaries

Let X be an n-dimensional compact C^∞ manifold with boundary $\partial X = X'$, X and X' provided with C^∞ vector bundles E and E'. We can assume that X is smoothly imbedded in an n-dimensional manifold $\widetilde{X}$ without boundary, provided with a vector bundle $\widetilde{E}$ such that $E = \widetilde{E}|_X$. We consider the algebra of (one-step) polyhomogeneous operators in the calculus of Boutet de Monvel

$$\begin{pmatrix} P_+ + G & K \\ & \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(E) \\ \oplus \\ C^\infty(E') \end{matrix} \longrightarrow \begin{matrix} C^\infty(E) \\ \oplus \\ C^\infty(E') \end{matrix}. \tag{2.1}$$

Here P is a pseudodifferential operator satisfying the transmission condition. The subscript '+' indicates that we are taking the truncation of P to X, i.e., the operator given by extending a function in $C^\infty(E)$ by zero to a function in $L^2(\widetilde{X}, \widetilde{E})$, applying P, and restricting the result to X°; the transmission property assures that this gives an element of $C^\infty(E)$. Moreover, G is a singular Green operator (s.g.o.), K a Poisson operator, T a trace operator, and S a pseudodifferential operator on X'. We assume that all are of order ν. Details on the calculus can e.g. be found in [G2].

As a first observation we note that for classical (one-step polyhomogeneous) ψdo's P of order ν on $\widetilde{X}$, the calculus of Boutet de Monvel requires $\nu \in \mathbb{Z}$ in order for the transmission condition to be satisfied at ∂X. The definition of the operators G, K and T (and, of course, S), however, extends readily to noninteger ν, so one can also ask for an extension of the canonical trace to operators of the form

$$\begin{pmatrix} G & K \\ T & S \end{pmatrix}, \tag{2.2}$$

with pseudodifferential part equal to zero and all elements of order $\nu \in \mathbb{R}$.

We next fix an auxiliary operator: We let P_1 be a second-order strongly elliptic differential operator in $\widetilde{E}$ with scalar principal symbol, and denote by $P_{1,D}$ its Dirichlet realization on X. By possibly shifting the operator by a constant, we can assume that P_1 and $P_{1,\mathrm{D}}$ have positive lower bound, so that the resolvents $Q_\lambda = (P_1 - \lambda)^{-1}$ and $R_\lambda = (P_{1,\mathrm{D}} - \lambda)^{-1}$ are defined for λ in a region

$$\Lambda = \{\, \lambda \in \mathbb{C} \mid \arg \lambda \in [\tfrac{\pi}{2} - \varepsilon, \tfrac{3\pi}{2} + \varepsilon] \text{ or } |\lambda| \le r(\varepsilon) \,\} \tag{2.3}$$

with suitable $\varepsilon > 0$ and $r(\varepsilon) > 0$. Moreover, we take a strongly elliptic second-order pseudodifferential operator S_1 on $C^\infty(X', E')$ and set $B = \begin{pmatrix} P_{1,D} & 0 \\ 0 & S_1 \end{pmatrix}$. Then

$$\begin{aligned} \operatorname{Tr}\left(\begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} (B-\lambda)^{-N} \right) \\ = \operatorname{Tr}_X \left((P_+ + G)(P_{1,D}-\lambda)^{-N} \right) + \operatorname{Tr}_{X'} \left(S(S_1-\lambda)^{-N} \right) \end{aligned} \tag{2.4}$$

is well-defined for $N > (n+\nu)/2$. The behavior of $\operatorname{Tr}_{X'} \left(S(S_1-\lambda)^{-N} \right)$ is well-known from the theory of ψdo's on manifolds without boundary, so we shall restrict the attention to

$$\operatorname{Tr}_X \left((P_+ + G)(P_{1,D}-\lambda)^{-N} \right) \text{ if } \nu \in \mathbb{Z}, \text{ and } \operatorname{Tr}_X \left(G(P_{1,D}-\lambda)^{-N} \right) \text{ if } \nu \in \mathbb{R} \setminus \mathbb{Z}. \tag{2.5}$$

We henceforth denote $P_+ + G = A$.

A natural candidate for an extension of the canonical trace is the functional $A \mapsto C_0(A, P_{1,\mathrm{D}})$ which associates with the ψdbo $A = P_+ + G$ the coefficient of $(-\lambda)^{-N}$ in (1.10), equal to the coefficient of s^0 in the Laurent expansion of $\zeta(A, P_{1,\mathrm{D}}, s)$ at $s = 0$ (1.14). We shall show that this is indeed a good choice at least in two cases:

(i) $A = G$ is a singular Green operator of noninteger order ν and class zero with vanishing pseudodifferential part;

(ii) $A = P_+ + G$ is of integer order ν with a pseudodifferential operator P of normal order 0 and G of class zero (we say that a ψdo symbol has normal order d when it is $O(\xi_n^d)$ at the boundary).

There are two conditions involved, namely on the class of the singular Green part and on the normal order of the pseudodifferential part in (ii). They are both natural, and related:

If G is effectively of class $r > 0$, i.e., if it can be written in the form $G = G_0 + \sum_{j=0}^{r-1} K_j \gamma_j$, where G_0 is a singular Green operator of class zero, $\gamma_j u = (\partial_n^j u)|_{X'}$, and the K_j are Poisson operators with $K_{r-1} \neq 0$, then G will not be bounded on $L^2(E)$, much less of trace class, even if its order is arbitrarily low, cf. e.g. [G2, Sect. 2.8]. As we are looking for a functional coinciding with the standard trace for operators of sufficiently low order, we shall exclude operators of positive class. An expansion analogous to (1.10) still exists, with the sum over $k \geq 0$ replaced by a sum over $k \geq k_0$ with a possibly negative starting index k_0. In this case, however, the coefficient c_0' will not in general coincide with the noncommutative residue, cf. [GSc, Remark 1.2].

For operators P having the transmission property, we find that $C_0(P_+, P_{1,\mathrm{D}})$ does indeed have a value modulo local terms that is independent of P_1 modulo local terms. The commutator $[P_+, P'_+]$ of two pseudodifferential operators P and P' truncated to X, however, will be of the form $P''_+ + G''$, where, in general, the singular Green operator G'' will be of class $r > 0$ unless both P and P' are of normal order ≤ 0, cf. [G2, Section 2.6]. Hence we can only hope to show the commutator property for this class.

Recall (cf. e.g. [G2, Lemma 1.3.1]) that any ψdo P having the transmission property at X' can be written as a sum

$$P = P^{(1)} + P^{(2)} + P^{(3)}, \tag{2.6}$$

where $P^{(1)}$ is a differential operator, $P^{(2)}$ is a ψdo whose symbol has normal order -1, and the symbol of $P^{(3)}$ has, near X', a large power of x_n as a factor to the left or right.

For a *differential* operator $P^{(1)}$, the expansion in (1.10) (with $A = P^{(1)}$) is valid without the second series over k; all terms are locally determined. Therefore $C_{-1}(P^{(1)}, P_{1,\mathrm{D}})$ vanishes, and $C_0(P^{(1)}, P_{1,\mathrm{D}})$ is local in this case, and so is $C_0(P^{(1)}, P_{1,\mathrm{D}}) - C_0(P^{(1)}, P_{2,\mathrm{D}})$ for another choice of auxiliary operator P_2. Thus $C_0(P^{(1)}, P_{1,\mathrm{D}})$ satisfies the first part of the requirement for being a quasi-trace (with value zero modulo local terms). As mentioned above, differential operators of positive normal order will be left out from the commutator considerations. So will operators of type $P^{(3)}$, since the structure is not preserved under differentiation of symbols.

At this point we can state a part of the results we will show in this paper:

Theorem 2.1. *Let $A = P_+ + G$ and $A' = P'_+ + G'$ be of orders ν resp. ν', with P resp. P' vanishing if ν resp. $\nu' \notin \mathbb{Z}$, and with G and G' of class 0. Let P_1 and P_2 be two auxiliary operators as explained above. Then*

(i)

$$C_0(A, P_{1,\mathrm{D}}) - C_0(A, P_{2,\mathrm{D}}) \text{ is locally determined;} \tag{2.7}$$

it vanishes if $\nu < -n$ or $\nu \notin \mathbb{Z}$, and otherwise depends solely on the terms of the first $\nu + n + 1$ homogeneity degrees in the symbols of $P_+ + G$, P_1 and P_2.

(ii) *If ν and $\nu' \in \mathbb{Z}$, assume that P and P' have normal order 0; if ν or $\nu' \in \mathbb{R} \setminus \mathbb{Z}$, assume that P and P' are zero. Then*

$$C_0([A, A'], P_{1,\mathrm{D}}) \text{ is locally determined;} \tag{2.8}$$

it vanishes if $\nu + \nu' < -n$ or $\nu + \nu' \notin \mathbb{Z}$, and otherwise depends solely on the terms of the first $\nu + \nu' + n + 1$ homogeneity degrees in the symbols of $P_+ + G$, $P'_+ + G'$ and P_1.

(Actually, the number of homogeneous terms entering from G is one step lower; see the statements in Section 3.)

The operators of the form $A = P_+ + G$ of integer order with G of class zero and P of normal order ≤ 0 form an algebra; Theorem 2.1 shows that $C_0(A, P_{1,\mathrm{D}})$ is a quasi-trace on this algebra as well as on the singular Green operators of class zero.

Similarly as in [GSc], our analysis is based on precise information about the resolvent. We therefore recall the structure of $R_\lambda = (P_{1,\mathrm{D}} - \lambda)^{-1}$: The Dirichlet problem

$$(P_1 - \lambda)u = f \text{ on } X, \quad \gamma_0 u = \varphi \text{ on } X', \tag{2.9}$$

is solved by a a row matrix

$$\begin{pmatrix} P_1 - \lambda \\ \gamma_0 \end{pmatrix}^{-1} = \begin{pmatrix} R_\lambda & K_\lambda \end{pmatrix} = \begin{pmatrix} Q_{\lambda,+} + G_\lambda & K_\lambda \end{pmatrix}, \text{ with } G_\lambda = -K_\lambda \gamma_0 Q_{\lambda,+}. \tag{2.10}$$

This is seen as follows: We denote by K_λ the Poisson operator solving the semi-homogeneous problem with $f = 0$, i.e., $K_\lambda \varphi$ is the solution u of the equations

$$(P_1 - \lambda)u = 0 \text{ on } X, \quad \gamma_0 u = \varphi \text{ on } X'.$$

The resolvent R_λ of $P_{1,\mathrm{D}}$ should solve the other semi-homogeneous problem, with $\varphi = 0$. With $Q_{\lambda,+}$ denoting the truncation to X of the resolvent Q_λ of P_1 on $\widetilde{X}$, one can easily check that $R_\lambda = Q_{\lambda,+} - K_\lambda \gamma_0 Q_{\lambda,+}$ solves the problem; it has the singular Green operator part $G_\lambda = -K_\lambda \gamma_0 Q_{\lambda,+}$.

We shall also study powers of the resolvent and write them in the form

$$R_\lambda^N = (Q_{\lambda,+} + G_\lambda)^N = (Q_\lambda^N)_+ + G_\lambda^{(N)}. \tag{2.11}$$

Note that

$$R_\lambda^N = \tfrac{1}{(N-1)!}\partial_\lambda^{N-1} R_\lambda, \quad Q_{\lambda,+}^N = \tfrac{1}{(N-1)!}\partial_\lambda^{N-1} Q_{\lambda,+}^N, \quad G_\lambda^{(N)} = \tfrac{1}{(N-1)!}\partial_\lambda^{N-1} G_\lambda; \tag{2.12}$$

this allows us to replace calculations for powers of R_λ by calculations for λ-derivatives.

We shall now start with the proof of Theorem 2.1. The case where $A = G$ is a singular Green operator of class zero and the pseudodifferential part vanishes can be attacked more directly and will be studied first.

3. Traces on the algebra of singular Green operators

Let G be a singular Green operator in E of order $\nu \in \mathbb{R}$ and class 0, and consider $GR_\lambda^N = GQ_{\lambda,+}^N + GG_\lambda^{(N)}$. In [GSc] we introduced the auxiliary variable $\mu = (-\lambda)^{\frac{1}{2}}$ and phrased the results in terms of μ instead of λ. In the present paper we shall keep λ as an index, but will often use μ in symbol computations. Some formulas from [GSc] and some immediate consequences are collected in Appendices A and B, to which we refer in the following.

As mentioned already, the proof in [GSc] of the expansion (1.10) for $A = G$ is straightforwardly modified to allow $\nu \notin \mathbb{Z}$. For an analysis of the coefficients, we apply a partition of unity $1 = \sum_{i=1}^{i_0} \theta_i$ subordinate to a cover of X by coordinate patches U_j, $j = 1, \dots, j_0$, with trivializations $\psi_j \colon E|_{U_j} \to V_j \times \mathbb{C}^{\dim E}$, $V_j \subset\subset \overline{\mathbb{R}}^n_+$, $U_j \cap \partial X$ mapped into $\partial \overline{\mathbb{R}}^n_+$, such that any two of the functions θ_{i_1} and θ_{i_2} are supported in one of the coordinate patches $U_{j(i_1,i_2)}$. Replacing G by $\sum_{i_1,i_2 \le i_0} \theta_{i_1} G \theta_{i_2}$, we find from $GQ_{\lambda,+}^N$ and $GG_\lambda^{(N)}$ a system of terms

$$G = \sum_{i_1,i_2 \le i_0} (\theta_{i_1} G \theta_{i_2} Q_{\lambda,+}^N + \theta_{i_1} G \theta_{i_2} G_\lambda^{(N)}), \tag{3.1}$$

where θ_{i_1} and θ_{i_2} are supported either in an interior coordinate patch (with closure in the interior) or a patch meeting the boundary, and we can study each term in the local coordinates. Since $\theta_{i_2} G_\lambda^{(N)}$ is strongly polyhomogeneous of order $-\infty$ when θ_{i_2} is supported in the interior, its kernel is smooth and is $O(\lambda^{-M})$ for any M. Hence the terms $\theta_{i_1} G \theta_{i_2} G_\lambda^{(N)}$ that are supported in the interior have traces that are $O(\lambda^{-M})$ for any M and can be disregarded in the following. For the terms $\theta_{i_1} G \theta_{i_2} Q_{\lambda,+}^N$ supported in the interior, $\theta_{i_1} G \theta_{i_2}$ is of order $-\infty$; they will be covered by Lemma 3.1 below, which gives:

$$\operatorname{Tr}(\theta_{i_1} G \theta_{i_2} Q_{\lambda,+}^N) = \operatorname{Tr}(\theta_{i_1} G \theta_{i_2})(-\lambda)^{-N} + O(\lambda^{-N-1/2}). \tag{3.2}$$

For the terms supported in patches meeting the boundary we can use the results of [GSc] in an accurate way.

Let us first recall some elements of the basic symbol calculus we use, namely the calculus of parameter-dependent ψdo's introduced in [GS1]. It will be used both in the $(n-1)$-dimensional setting, relevant for operators on X', and in the n-dimensional setting, relevant for operators on $\widetilde{X}$.

For $m \in \mathbb{R}$, the usual pseudodifferential symbol space $S^m(\mathbb{R}^{n'} \times \mathbb{R}^{n'})$ consists of the functions $s(x,\xi) \in C^\infty(\mathbb{R}^{n'} \times \mathbb{R}^{n'})$ satisfying estimates

$$|D_x^\beta D_\xi^\alpha s(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|},$$

for all $\alpha, \beta \in \mathbb{N}^{n'}$. (We denote $(1+|\xi|^2)^{\frac{1}{2}} = \langle \xi \rangle$, $\{0,1,2,\dots\} = \mathbb{N}$.) The parameter-dependent version we shall use here is $S^{m,d,s}(\mathbb{R}^{n'} \times \mathbb{R}^{n'}, \Gamma)$, for $m \in \mathbb{R}$, d and $s \in \mathbb{Z}$, and with Γ denoting a sector in $\mathbb{C} \setminus \{0\}$; it was introduced in [G3], where a detailed account can be found. Here $S^{m,d,s}(\mathbb{R}^{n'} \times \mathbb{R}^{n'}, \mathbb{R}_+)$ is defined as the space of C^∞ functions $f(x,\xi,\mu)$ on $\mathbb{R}^{n'} \times \mathbb{R}^{n'} \times \mathbb{R}_+$ such that, with $z = 1/\mu$,

$$\partial_z^j [z^d |(\xi, 1/z)|^{-s} f(x,\xi,1/z)] \in S^{m+j}(\mathbb{R}^{n'} \times \mathbb{R}^{n'}),$$
with symbol estimates uniform in z for $z \le 1$.

For more general Γ, the estimates have to hold on each ray in Γ, uniformly in closed subsectors of Γ. When the symbols moreover are holomorphic in $\mu \in \Gamma^\circ$ (just for $|(\mu,\xi')| \ge \varepsilon$ with some ε depending on the closed subsector), we speak of holomorphic symbols. The indication $\mathbb{R}^{n'} \times \mathbb{R}^{n'}$ (and Γ) will often be omitted. We say that $f(x,\xi,\mu)$ is (weakly) polyhomogeneous in $S^{m,d,s}(\Gamma)$, when there is a sequence of symbols f_j in $S^{m-j,d,s}(\Gamma)$, homogeneous in (ξ,μ) of degree $m-j+d+s$ for $|\xi| \ge 1$, such that $f - \sum_{j<J} f_j$ is in $S^{m-J,d,s}(\Gamma)$ for all $J \in \mathbb{N}$ (strongly polyhomogeneous if the homogeneity holds for $|(\xi,\mu)| \ge 1$, with appropriate remainder estimates).

These spaces $S^{m,d,s}$ are a generalization of the spaces $S^{m,d}$ introduced in [GS1] (in fact, $S^{m,d}(\Gamma)$ equals the space of holomorphic symbols in $S^{m,d,0}(\Gamma)$); they satisfy

$$\begin{aligned} S^{m,d,s} &\subset S^{m+s,d,0} \cap S^{m,d+s,0} \text{ if } s \le 0, \\ S^{m,d,s} &\subset S^{m+s,d,0} + S^{m,d+s,0} \text{ if } s \ge 0, \end{aligned} \tag{3.3}$$

by [GS1, Lemma 1.13]. The notation with the third upper index spares us from keeping track of such intersections and sums in general calculations. (It was used systematically in [G3], [G5], [G6].)

When $f \in S^{m,d,0}$, there is an expansion in decreasing powers of μ starting with μ^d:

$$f(x,\xi,\mu) = \sum_{0 \le k < N} \mu^{d-k} f_{(k)}(x,\xi) + O(\mu^{d-N} \langle \xi \rangle^{m+N}), \quad \text{any } N; \tag{3.4}$$

in fact the N'th remainder is in $S^{m+N,d-N,0}$. Such expansions play an important role in the proof of the following fact: When $F = \mathrm{OP}(f(x,\xi,\mu))$ has a polyhomogeneous symbol in $S^{m,d,0}$ such that all terms are integrable with respect to ξ, then by [GS1, Th. 2.1], the kernel of F has an asymptotic expansion on the diagonal:

$$K(F,x,x,\mu) \sim \sum_{j \in \mathbb{N}} c_j(x) \mu^{m+d+n'-j} + \sum_{k \in \mathbb{N}} (c'_k(x) \log \mu + c''_k(x)) \mu^{d-k}; \tag{3.5}$$

here the coefficients $c_j(x)$ and $c'_k(x)$ with $k = -m + j - n'$ are local, determined from the strictly homogeneous version of the j'th homogeneous term in the symbol of F, whereas the $c''_k(x)$ depend on the full operator (are global). If $m \notin \mathbb{Z}$, all c'_k are zero. When F is a ψdo family on $\widetilde{X}$ (or on X'), the trace expansion for F is found by establishing expansions (3.5) for components of F in local coordinate systems, carrying them back to $\widetilde{X}$ and integrating the fibrewise traces over $\widetilde{X}$; this leads to a similar expansion of $\operatorname{Tr} F$:

$$\operatorname{Tr}(F) \sim \sum_{j\in\mathbb{N}} c_j \mu^{m+d+n'-j} + \sum_{k\in\mathbb{N}} (c'_k \log\mu + c''_k)\mu^{d-k}. \tag{3.6}$$

Of course, such a calculation depends on the choice of partitions of unity and local coordinates, but it is useful for qualitative information. The final result (3.6) is independent of the choices, in the sense that the collected coefficient of each power or log-power is so (for $k = j - m - n' \in \mathbb{N}$, $c_j + c''_k$ is independent of the choices).

This is the basis for the trace expansions we shall show. For purely ψdo terms on X, the expansions will be established relative to $\widetilde{X}$, but the integration of the kernels will be restricted to X. For all the other types of operators, the main idea is to reduce the considerations to ψdo's over the boundary X'. In all cases the point is to show that the resulting symbols belong to suitable $S^{m,d,0}$-spaces such that (3.5) can be established.

Since we are particularly interested in the nonlocal coefficient c''_0, we have to be very careful with remainders and smoothing terms. Recall from [GS1, Prop. 1.21] that when an operator F with symbol in $S^{m,d,0}$ is of order $-\infty$ (i.e., lies in $S^{-\infty,d,0}$), then it has a C^∞ kernel $K(F,x,y,\mu)$ with an expansion in C^∞ terms

$$K(F,x,y,\mu) = \sum_{0\le l<L} K_l(x,y)\mu^{d-l} + K'_L(x,y,\mu), \tag{3.7}$$

for all L, with K'_L and its (x,y)-derivatives being $O(\langle\mu\rangle^{d-L})$. Moreover, since $\partial_z^j(z^d F) = (-\mu^2\partial_\mu)^j(\mu^{-d}F)$ has symbol in $S^{-\infty,0,0}$ for all j, there are expansions

$$K((-\mu^2\partial_\mu)^j(\mu^{-d}F),x,y,\mu) = \sum_{0\le l<L} K_l^{(j)}(x,y)\mu^{-l} + K_L^{(j)\prime}(x,y,\mu), \tag{3.8}$$

for all L, with $K_L^{(j)\prime}$ and its (x,y)-derivatives being $O(\langle\mu\rangle^{-L})$. Conversely, these kernel properties characterize the operators F with symbol in $S^{-\infty,d,0}$ (as is seen straightforwardly from the definition).

It follows from (3.3) that $S^{-\infty,d,s} = S^{-\infty,d+s,0}$. In the proof that a polyhomogeneous symbol lies in a symbol space $S^{m,d,s}$ one shows the symbol properties for the homogeneous terms and treats the remainders by establishing the kernel expansionss (3.8) (with d replaced by $d+s$) with smoother coefficients and a higher number of terms, the higher the index of the remainder is taken.

The sector Γ used in the present work is

$$\Gamma = \{\, \mu \in \mathbb{C}\setminus\{0\} \mid |\arg\mu| \le \tfrac{\pi}{4} + \tfrac{\varepsilon}{2} \,\}, \tag{3.9}$$

for some $\varepsilon > 0$, since $\lambda = -\mu^2$ runs in Λ defined in (2.3). *All symbols to be considered in this paper will be holomorphic on* Γ°; this fact will not be mentioned explicitly each time.

One of the technical points in [GSc] was that in compositions of two operators, the right-hand factor is taken in y-form and the resulting symbol is considered in (x, y)-form so that one does not need to consider the asymptotic expansion in the usual symbol composition formula (cf. e.g. (3.31) below). However, the passage from one of these forms to another can induce low-order errors. These have to be dealt with carefully in the present situation, where nonlocal contributions are in focus. Also for this reason, the following analysis of the effect of smoothing operators is important.

Smoothing parts of Q_λ and G_λ are easy to deal with, since they have smooth kernels that are $O(\lambda^{-M})$ for any M. For compositions containing smoothing parts of G, we recall that an s.g.o. of order $-\infty$ is simply an operator with C^∞ kernel on X. So is P_+ when P is of order $-\infty$. Such operators enter as follows (for $G^\pm$, cf. (A.14)):

Lemma 3.1. *Let G be an operator with C^∞ kernel, compactly supported in $\overline{\mathbb{R}}^n_+ \times \overline{\mathbb{R}}^n_+$. Then the traces of $GQ^N_{\lambda,+}$, $GG^{(N)}_\lambda$ and $GG^\pm(Q^N_\lambda)$ have expansions $\sum_{j\ge 0} c''_j \mu^{-2N-j}$, where*

$$\text{(3.10)}\qquad \begin{aligned} &\text{(i)} \quad c''_0 = \operatorname{Tr} G \text{ in the expansion of } \operatorname{Tr}(GQ^N_{\lambda,+}),\\ &\text{(ii)} \quad c''_0 = 0 \text{ in the expansions of } GG^{(N)}_\lambda \text{ and } GG^\pm(Q^N_\lambda). \end{aligned}$$

Proof. First consider the cases in (ii); here we use that the symbol of the λ-dependent factor has the structure described in Lemma A.4 with $\mu^2 = -\lambda$. Consider a term $G_{J,j,j'} = \mathrm{OPG}(g_{J,j,j'}(x', \xi', \xi_n, \eta_n, \mu))$ with $g_{J,j,j'} = s_{J,j,j'}(\kappa + i\xi_n)^{-j}(\kappa' - i\eta_n)^{-j'}$ (κ and κ' taking values κ^+ or κ^-), $s_{J,j,j'} \in S^{0,0,j+j'-1-2N-J}$; its kernel is

$$\begin{aligned} K(G_{J,j,j'}, x, y, \mu) &= \int e^{i(x'-y')\cdot\xi' + ix_n\xi_n - iy_n\eta_n} g_{J,j,j'}(x', \xi', \xi_n, \eta_n, \mu)\, đ\xi đ\eta_n\\ &= c_j c_{j'} \int e^{i(x'-y')\cdot\xi'} s_{J,j,j'}(x', \xi', \mu) x_n^{j-1} y_n^{j'-1} e^{-\kappa x_n - \kappa' y_n}\, đ\xi', \end{aligned}$$

since $\mathcal{F}^{-1}_{\xi_n\to x_n}(\kappa + i\xi_n)^{-j} = c_j x_n^{j-1} e^{-\kappa x_n}$. Now

$$\begin{aligned} K(GG_{J,j,j'}, x, y, \mu) &= \int_{\mathbb{R}^n_+} K(G, x, z) K(G_{J,j,j'}, z, y, \mu)\, dz,\\ \operatorname{Tr}(GG_{J,j,j'}) &= \int_{\mathbb{R}^n_+} \operatorname{tr} \int_{\mathbb{R}^n_+} K(G, x, z) K(G_{J,j,j'}, z, x, \mu)\, dz dx. \end{aligned}$$

To evaluate the latter, we insert a Taylor expansion of $K(G, x, z)$ w.r.t. (x_n, z_n):

$$K(G, x, z) = \sum_{m,m'<M} b_{m,m'}(x', z') x_n^m z_n^{m'} + x_n^M z_n^M b'_M(x, z),$$

and use that

$$\begin{aligned} &\int_0^\infty x_n^{m+j-1} e^{-\kappa x_n}\, dx_n = c'_{m+j}\kappa^{-m-j},\\ &|\int_0^\infty f(x_n) x_n^{M+j-1} e^{-\kappa x_n}\, dx_n| \le \sup |f(x_n) e^{-\kappa x_n/2} \int_0^\infty x_n^{M+j-1} e^{-\kappa x_n/2}\, dx_n|\\ &\qquad \le C\kappa^{-M-j}. \end{aligned}$$

This gives when integrations in (x_n, z_n) are performed:

$$\operatorname{Tr}(GG_{J,j,j'}) = \sum_{m,m'<M} c_{jj'mm'} \int \operatorname{tr} b_{m,m'}(x',z')\kappa^{-m-j'}(\kappa')^{-m'-j} e^{i(z'-x')\cdot\xi'} s_{J,j,j'}(z',\xi',\mu)\, đ\xi' dz' dx' + \int O(\langle(\xi',\mu)\rangle^{-2M-1-2N-J})\, đ\xi'.$$

Each term in the sum over m, m' is the trace of a ψdo $S_{J,j,j',m,m'}$ on $\mathbb{R}^{n-1}$ composed of the operator with kernel $c_{jj'mm'}b_{m,m'}(x',y')$ (smooth compactly supported) and the operator with symbol

$$\kappa(x',\xi',\mu)^{-m-j'}\kappa'(x',\xi',\mu)^{-m'-j} s_{J,j,j'}(x',\xi',\mu) \in S^{0,0,-m-m'-1-2N-J};$$

so $S_{J,j,j',m,m'}$ has symbol in $S^{-\infty,0,-m-m'-1-2N-J} = S^{-\infty,-m-m'-1-2N-J,0}$ and consequently a trace expansion as in (3.7) with $d \le -2N-1$ in all cases. The remainder is $O(\mu^{-2M-1-2N-J+n-1})$. Since M can be taken arbitrarily large, this shows (ii).

For (i), let $\widetilde{G}$ be an operator on $\mathbb{R}^n$ with compactly supported C^∞ kernel such that $G = r^+\widetilde{G}e^+$ (its kernel can be constructed by extending the kernel of G smoothly across $x_n = 0$ and $y_n = 0$). Then

$$GQ^N_{\lambda,+} = \widetilde{G}_+ Q^N_{\lambda,+} = (\widetilde{G}Q^N_\lambda)_+ - G^+(\widetilde{G})G^-(Q^N_\lambda).$$

Here $G^+(\widetilde{G})G^-(Q^N_\lambda)$ is of the type we considered above, having a trace expansion with $d \le -2N-1$. For $(\widetilde{G}Q^N_\lambda)_+$ we use the result known from [GS1] (cf. also [G4, Th. 1.3]) that the kernel of $\widetilde{G}Q^N_\lambda$ has an expansion on the diagonal

$$K(\widetilde{G}Q^N_\lambda, x, x, \mu) \sim \sum_{j\ge 0} c''_j(x)\mu^{-2N-j}, \text{ with } c''_0(x) = K(\widetilde{G}, x, x). \tag{3.11}$$

The expansion of $\operatorname{Tr}(\widetilde{G}Q^N_\lambda)_+$ is found by integrating the matrix trace of (3.11) over $\mathbb{R}^n_+$, where $K(\widetilde{G},x,x) = K(G,x,x)$, so (i) follows, with $c''_0 = \int_{\mathbb{R}^n_+} \operatorname{tr} K(G,x,x)\,dx = \operatorname{Tr} G$. $\square$

As already mentioned, this lemma takes care of the interior parts of $GQ^N_{\lambda,+}$, cf. (3.1), (3.2).

For the parts $\theta_{i_1}G\theta_{i_2}G^{(N)}_\lambda$ at the boundary, we can use Proposition B.3 directly; it shows that they contribute only locally to the coefficient of λ^{-N}.

It remains to consider terms $\theta_{i_1}G\theta_{i_2}Q^N_{\lambda,+}$ in patches intersecting the boundary. Here we can draw on the fine analysis of such terms made in [GSc] to find the contribution to c'_0 in (1.12); it can be used also to isolate the contribution to c''_0 modulo local terms. This will give not only a proof of (2.7) but also a "value" (modulo local terms) of the constant.

To simplify the presentation, we can unify the treatment of the coordinate patches by spreading out the images $V_{j(i_1,i_2)}$ of the sets $U_{j(i_1,i_2)}$ by linear translations in the x'-variable, to obtain sets V'_{i_1,i_2} with positive distance (in the x'-direction) from one another.

Then the sum of the localized operators $\theta_{i_1} G\theta_{i_2}$ acts in $\overline{\mathbb{R}}^n_+ \times \mathbb{C}^{\dim E}$; we shall denote it by G again or, if a distinction from the original G is needed, by $\underline{G}$. R_λ is likewise considered in these coordinates and may be denoted $\underline{R}_\lambda$ if needed for precision.

Recall that when G is defined on $\overline{\mathbb{R}}^n_+$ from a symbol $g(x',\xi',\xi_n,\eta_n)$ with Laguerre expansion (B.15), then the operator $\operatorname{tr}_n G$ is defined as the ψdo on $\mathbb{R}^{n-1}$ with symbol

$$\operatorname{tr}_n g(x',\xi') = \int g(x',\xi',\xi_n,\xi_n)\, đ\xi_n = \sum_{j\in\mathbb{N}} d_{jj}(x',\xi').$$

Recall also (cf. e.g. [G1]) that a singular Green operator on $\overline{\mathbb{R}}^n_+$ with compact x'-support is trace-class, when its order is $< -n+1$. Then $\operatorname{tr}_n G$ is trace-class on $\mathbb{R}^{n-1}$ and $\operatorname{Tr}_{\mathbb{R}^n} G = \operatorname{Tr}_{\mathbb{R}^{n-1}}(\operatorname{tr}_n G)$.

We shall now apply the result of Proposition B.5; that $\operatorname{tr}_n(GQ^N_{\lambda,+}) = \widetilde{S}_0 + \widetilde{S}_1$, with $\widetilde{S}_0 = \mathrm{OP}'(\operatorname{tr}_n g(x',\xi')\alpha^{(N)}(x',\xi',\mu))$ (cf. (A.12)) and $\widetilde{S}_1$ having symbol in $S^{\nu+1,-2N-1,0} \cap S^{\nu-2N,0,0}$. This result was used in [GSc] to pinpoint the first logarithmic coefficient $\tilde{c}_0'$ in (1.10), using that only $\widetilde{S}_0$ contributes to it. But the fact that $\widetilde{S}_1$ has d-index $-1-2N$ also implies that it contributes only locally to $\tilde{c}_0''$, so nonlocal contributions to $\tilde{c}_0''$ come entirely from $\widetilde{S}_0$.

In the localized situation, denote $\operatorname{tr}_n G = \overset{\approx}{G}$ and denote its symbol $\operatorname{tr}_n g = \overset{\approx}{g}$, expanded in homogeneous terms $\overset{\approx}{g} \sim \sum_{j\ge 0} \overset{\approx}{g}_{\nu-j}$.

We shall use the notion of a regularized integral (or finite part integral) $⨍ f(\xi)\, đ\xi$ as in Lesch [L], [G4], for polyhomogeneous functions $f(\xi)$ of $\xi \in \mathbb{R}^{n'}$ (it will be used with $n' = n-1$ or n): $⨍ f(\xi)\, đ\xi$ is the constant term in the expansion of $\int_{|\xi|\le\mu} f(\xi) đ\xi$ into powers of μ and $\log\mu$. In more detail:

When $f(\xi)$ is integrable in ξ, $⨍ f(\xi)\, đ\xi$ is the usual integral $\int_{\mathbb{R}^{n'}} f(\xi)\, đ\xi$. When $f_{\nu-j}(\xi)$ is homogeneous of order $\nu-j$ in ξ for $|\xi|\ge 1$, then

$$\int_{|\xi|\le\mu} f_{\nu-j}\, đ\xi = \int_{|\xi|\le 1} f_{\nu-j}\, đ\xi + \int_{1\le|\xi|\le\mu} f_{\nu-j}\, đ\xi, \tag{3.12}$$

where a calculation in polar coordinates gives

$$\begin{aligned} \int_{1\le|\xi|\le\mu} f_{\nu-j}(\xi)\, đ\xi &= \int_1^\mu r^{\nu-j+n'-1}\, dr \int_{|\xi|=1} f_{\nu-j}(\xi)\, đS(\xi) \\ &= \begin{cases} c_j\big(\frac{\mu^{\nu-j+n'}}{\nu-j+n'} - \frac{1}{\nu-j+n'}\big) & \text{if } \nu-j+n' \ne 0, \\ c_j \log\mu & \text{if } \nu-j+n' = 0; \end{cases} \\ c_j &= \int_{|\xi|=1} f_{\nu-j}(\xi)\, đS(\xi). \end{aligned} \tag{3.13}$$

Here,

$$⨍ f_{\nu-j}(\xi)\, đ\xi = \int_{|\xi|\le 1} f_{\nu-j}\, đ\xi + \begin{cases} -\frac{1}{\nu-j+n'} c_j & \text{if } \nu-j+n' \ne 0, \\ 0 & \text{if } \nu-j+n' = 0; \end{cases} \tag{3.14}$$

this is consistent with the integrable case. (Such calculations were basic in the proof of [GS1, Th. 2.1].) More generally, when $f(x,\xi)$ is a classical symbol of order ν, $f(x,\xi)\sim\sum_{j\in\mathbb{N}} f_{\nu-j}(x,\xi)$, then there is an asymptotic expansion for $\mu\to\infty$,

$$\int_{|\xi|\le\mu} f(x,\xi)\,đ\xi \sim \sum_{j\in\mathbb{N},j\ne\nu+n'} f_j(x)\mu^{n'+\nu-j} + f_0'(x)\log\mu + f_0''(x), \tag{3.15}$$

and we set

$$⨍ f(x,\xi)\,đ\xi = f_0''(x). \tag{3.16}$$

(This is related to Hadamard's definition of the finite part — *partie finie* — of certain integrals [H, p. 184ff.].) In view of (3.12)–(3.14), we have the precise formula:

$$\begin{aligned} ⨍ f(x,\xi)\,đ\xi = \sum_{0\le j\le\nu+n'} \Big(\int_{|\xi|\le 1} f_{\nu-j}(x,\xi)\,đ\xi - \tfrac{1-\delta_{\nu+n',j}}{\nu+n'-j}\int_{|\xi|=1} f_{\nu-j}\,đS(\xi)\Big)& \\ + \int_{\mathbb{R}^{n'}} \big(f - \sum_{j\le\nu+n'} f_{\nu-j}\big)\,đ\xi,& \end{aligned} \tag{3.17}$$

where $\delta_{r,s}$ is the Kronecker delta. (One can replace the sum over $j\le\nu+n'$ by the sum over $j\le J$ for any $J\ge\nu+n'$.)

Remark 3.2. Let $F=\mathrm{OP}(f(x,\xi))$ be the ψdo associated with the above symbol. It was shown by Lesch [L, Section 5] that the density $\omega(F)=⨍ f(x,\xi)\,đ\xi dx$ is invariant under the change of the symbol induced by diffeomorphisms of open sets, when $\nu\notin\mathbb{Z}$ or $\nu<-n'$. Moreover, it was observed in [G4] that the proof of [L] extends to the cases where $\nu\in\mathbb{Z}$ and f has a parity that fits with the dimension n', cf. (1.4)–(1.5)ff. So in all these cases, $\omega(F)$ depends only on F, not on the representation of its symbol in local coordinates. As a consequence, we can consider this density also for ψdo's having the mentioned properties on a manifold M of dimension n'. Note that when M is compact, $\int_M ⨍ \operatorname{tr} f(x,\xi)\,đ\xi dx = \mathrm{Tr}(F)$ if $\nu<-n'$.

The notation is used in the following with $n'=n-1$. It was shown in [G4, Th. 1.3] (by working out [GS1, Th. 2.1] explicitly in this case) that when $f(x',\xi')$ is the symbol of a classical ψdo F on $\mathbb{R}^{n-1}$, and $S=\mathrm{OP}'(s(x',\xi'))$ is an auxiliary second-order uniformly elliptic operator with no principal eigenvalues on $\mathbb{R}_-$, then the kernels of $F(S+\mu^2)^{-N}$ and $\mathrm{OP}'(f(x',\xi')(s(x',\xi')+\mu^2)^{-N})$ with $N>(\nu+n-1)/2$ have diagonal expansions of the form

$$K(x',x',\mu)\sim\sum_{j\ge0} b_j(x')\mu^{n-1+\nu-j-2N} + \sum_{k\ge0}(b_k'(x')\log\mu + b_k''(x'))\mu^{-k-2N}, \tag{3.18}$$

where the b_k' are 0 if $\nu\notin\mathbb{Z}$, and

$$\begin{aligned} b_{\nu+n-1}(x') + b_0''(x') &= ⨍ f(x',\xi')\,đ\xi' + \text{ local terms, if } \nu \text{ is integer} \ge 1-n,\\ b_0''(x') &= ⨍ f(x',\xi')\,đ\xi' \text{ if } \nu<1-n \text{ or } \nu\notin\mathbb{Z}. \end{aligned} \tag{3.19}$$

Moreover, if $\nu \in \mathbb{Z}$ and f has a parity that fits with the dimension $n-1$, then $b_{\nu+n-1}(x') = 0$ and the second formula in (3.19) holds. (Furthermore, the sum over k in (3.18) for the operators $F(S+\mu^2)^{-N}$ and $\mathrm{OP}'(f(s+\mu^2)^{-N})$ skips the terms where $k/2 \notin \mathbb{Z}$, but we need to refer to the general expansion below. In parity cases, the b_j are zero for $\nu + n - 1 - j$ even, and the b'_k are zero.) What we shall show now is that $\alpha^{(N)}$ plays to a large extent the same role as $(s+\mu^2)^{-N}$. We shall deal with the special cases where $\nu < 1-n$ or $\nu \notin \mathbb{Z}$ here, whereas parity cases will be discussed later, around Theorem 3.15.

Theorem 3.3. *One has for $\widetilde{S}_0 = \mathrm{OP}'(\widetilde{\widetilde{g}}\alpha^{(N)})$, when $N > (n-1+\nu)/2$: The kernel of $\widetilde{S}_0$ has an expansion on the diagonal:*

$$K(\widetilde{S}_0, x', x', \mu) \sim \sum_{j\geq 0} a_j(x')\mu^{n-1+\nu-j-2N} + \sum_{k\geq 0}(a'_k(x')\log\mu + a''_k(x'))\mu^{-k-2N}, \tag{3.20}$$

where the terms $a_j(x')$, and $a'_k(x')$ for $k = \nu + n - 1 - j$, depend on the first $j+1$ homogeneous terms in the symbols $\widetilde{\widetilde{g}}(x',\xi')$ and $\alpha^{(N)}(x',\xi',\mu)$; the $a''_k(x')$ are global.

If $\nu < -n+1$ or $\nu \in \mathbb{R}\setminus\mathbb{Z}$, then

$$a''_0(x') = {⨍} \widetilde{\widetilde{g}}(x',\xi')\, đ\xi', \tag{3.21}$$

whereas if ν is an integer $\geq -n+1$,

$$a_{\nu+n-1}(x') + a''_0(x') = {⨍} \widetilde{\widetilde{g}}(x',\xi')\, đ\xi' + \text{ local terms}, \tag{3.22}$$

where the local terms depend on the first $\nu+n$ homogeneous terms in the symbols of $\widetilde{\widetilde{g}}$ and $\alpha^{(N)}$. The a'_k vanish if $\nu \notin \mathbb{Z}$.

Proof. The existence of the expansion is assured by [GS1, Th. 2.1], since $\widetilde{S}_0$ has a weakly polyhomogeneous symbol in $S^{\nu,-2N,0} \cap S^{\nu-2N,0,0}$, so the main point is to prove the formulas for a''_0 resp. $a_{\nu+n-1} + a''_0$. Recall that

$$K(\widetilde{S}_0, x', x', \mu) = \int_{\mathbb{R}^{n-1}} \widetilde{\widetilde{g}}(x',\xi')\alpha^{(N)}(x',\xi',\mu)\, đ\xi'.$$

First consider $\widetilde{\widetilde{g}}' = \widetilde{\widetilde{g}} - \sum_{j\leq \nu+n-1} \widetilde{\widetilde{g}}_{\nu-j}$, it is of order $\nu - J < -n+1$ where $J = \max\{n+[\nu], 0\}$, hence integrable in ξ'. Since

$$\alpha^{(N)} = \mu^{-2N} + \alpha_1^{(N)}(x',\xi',\mu)$$

with $\alpha_1^{(N)} \in S^{0,-2N,0} \cap S^{1,-2N-1,0}$ (cf. (A.13)ff.), $\widetilde{\widetilde{g}}'\alpha_1^{(N)}$ is in $S^{\nu-J,-2N,0} \cap S^{1+\nu-J,-2N-1,0}$, and we can write

$$K(\mathrm{OP}'(\widetilde{\widetilde{g}}'\alpha^{(N)}), x', x', \mu) = \mu^{-2N} \int_{\mathbb{R}^{n-1}} \widetilde{\widetilde{g}}'\, đ\xi' + \int_{\mathbb{R}^{n-1}} \widetilde{\widetilde{g}}'\alpha_1^{(N)}\, đ\xi'. \tag{3.23}$$

By [GS1, Th. 2.1], the last integral has an expansion as in (3.18) with ν replaced by $\nu - J$ and the sum over k starting with $k = 1$. This expansion has no term with μ^{-2N}, so the only contribution from $\tilde{\tilde{g}}'$ to the coefficient of μ^{-2N} is $\int \tilde{\tilde{g}}'\, đ\xi = ⨍ \tilde{\tilde{g}}'\, đ\xi$. This ends the proof in the case $\nu < -n + 1$, where $\tilde{\tilde{g}} = \tilde{\tilde{g}}'$.

For the homogeneous terms $\tilde{\tilde{g}}_{\nu-j}$, we use an analysis as in [GS1, Th. 2.1]: Let $\mu \in \mathbb{R}_+$ and write the contribution from $\tilde{\tilde{g}}_{\nu-j}$ as

$$\int_{|\xi'|\ge\mu} \tilde{\tilde{g}}_{\nu-j}(x',\xi')\alpha^{(N)}(x',\xi',\mu)\, đ\xi' + \int_{|\xi'|\le\mu} \tilde{\tilde{g}}_{\nu-j}(x',\xi')\alpha^{(N)}(x',\xi',\mu)\, đ\xi'.$$

The first integral gives a local term $\tilde{a}_j(x')\mu^{n-1+\nu-j-2N}$, since the integrand is homogeneous in (ξ',μ) of degree $\nu - j - 2N$. Note that when $\nu \in \mathbb{R}\setminus\mathbb{Z}$, the power cannot be $-2N$. The second integral may be written

$$\int_{|\xi'|\le\mu} \tilde{\tilde{g}}_{\nu-j}\alpha^{(N)}\, đ\xi' = \mu^{-2N}\int_{|\xi'|\le\mu} \tilde{\tilde{g}}_{\nu-j}(x',\xi')\, đ\xi' + \int_{|\xi'|\le\mu} \tilde{\tilde{g}}_{\nu-j}\alpha_1^{(N)}\, đ\xi'. \tag{3.24}$$

Here $\int_{|\xi'|\le\mu} \tilde{\tilde{g}}_{\nu-j}\, đ\xi'$ is calculated as in (3.12)–(3.14), contributing $⨍ \tilde{\tilde{g}}_{\nu-j}\, đ\xi'$ to the coefficient of μ^{-2N}. For the last integral in (3.24) we observe that since $\tilde{\tilde{g}}'\alpha_1^{(N)}$ is in $S^{\nu-J,-2N,0} \cap S^{1+\nu-J,-2N-1,0}$, with d-index $\le -2N-1$, the consideration of this integral in the proof of [GS1, Th. 2.1] shows that it contributes only locally to the coefficient of μ^{-2N}, and not at all if $\nu \notin \mathbb{Z}$. $\square$

Remark 3.4. Similar statements hold when $\tilde{\tilde{g}}$ is given in (x',y')-form; then $\tilde{\tilde{g}}(x',\xi')$ is replaced by $\tilde{\tilde{g}}(x',x',\xi')$ in (3.21) and (3.22). The extension to this case is carried out as in [G4, Remark 1.4].

In view of the information on general compositions recalled before the theorem, we have in particular:

Corollary 3.5. *Let $S = \mathrm{OP}'(s(x',\xi'))$ be a second-order uniformly elliptic operator on $\mathbb{R}^{n-1}$ with no principal eigenvalues on $\mathbb{R}_-$. Comparing the expansion* (3.18) *for the kernel of $\tilde{\tilde{G}}(S+\mu^2)^{-N}$ with the expansion* (3.20) *for the kernel of $\tilde{S}_0$, one has that*

$$\begin{aligned} a_0''(x') &= b_0''(x') + \text{ local terms, if } \nu \text{ is integer} \ge 1-n,\\ a_0''(x') &= b_0''(x') \text{ if } \nu < 1-n \text{ or } \nu \notin \mathbb{Z}. \end{aligned} \tag{3.25}$$

We then find:

Theorem 3.6. *Let G be a singular Green operator on X of order $\nu \in \mathbb{R}$ and class* 0.

1° *Let $g(x',\xi,\eta_n)$ be the symbol of the operator in a localization to $\overline{\mathbb{R}}^n_+$ as described above. Then*

$$C_0(G, P_{1,\mathrm{D}}) = \int ⨍ \operatorname{tr}(\operatorname{tr}_n g)(x',\xi')\, đ\xi' dx' + \text{ local terms}; \tag{3.26}$$

it holds with vanishing local terms if $\nu < -n+1$ *or* $\nu \in \mathbb{R}\setminus\mathbb{Z}$.

2° *Let* P_1 *and* P_2 *be two choices of auxiliary strongly elliptic operators. Then*

$$C_0(G, P_{1,\mathrm{D}}) - C_0(G, P_{2,\mathrm{D}}) \tag{3.27}$$

is locally determined; it vanishes if $\nu < -n+1$ *or* $\nu \in \mathbb{R}\setminus\mathbb{Z}$.

Proof. Consider G and $Q^N_{\lambda,+}$ carried over to $\overline{\mathbb{R}}^n_+$. Here we have

$$\operatorname{Tr}(GQ^N_{\lambda,+}) = \operatorname{Tr}_{\mathbb{R}^{n-1}} \operatorname{tr}_n(GQ^N_{\lambda,+}) = \operatorname{Tr}_{\mathbb{R}^{n-1}} \widetilde{S}_0 + \operatorname{Tr}_{\mathbb{R}^{n-1}} \widetilde{S}_1, \tag{3.28}$$

as in Proposition B.5. Since $\widetilde{S}_1$ has symbol in $S^{\nu+1,-2N-1,0} \cap S^{\nu-2N,0,0}$, it has a diagonal kernel expansion as in (3.18) with the sum over k starting with $k=1$, so the coefficient of μ^{-2N} is locally determined (vanishing if $\nu \in \mathbb{R}\setminus\mathbb{Z}$ or $\nu < 1-n$). For $\widetilde{S}_0$, we have the diagonal kernel expansion from Theorem 3.3. We integrate the matrix traces in x', and we add on $\operatorname{Tr}(GG^{(N)}_\lambda)$, which is described in Proposition B.3. This gives a trace expansion as in (1.10), with coefficient of $(-\lambda)^{-N} = \mu^{-2N}$ equal to

$$\int ⨍ \operatorname{tr}\operatorname{tr}_n g \, đ\xi' dx' + \text{ local terms}$$

(no local terms if $\nu \in \mathbb{R}\setminus\mathbb{Z}$ or $\nu < 1-n$); this shows 1°.

Now if P_1 is replaced by P_2 in these calculations, 1° implies that the contribution to the coefficient of μ^{-2N} is modified only in the local terms, vanishing if $\nu < -n+1$ or $\nu \in \mathbb{R}\setminus\mathbb{Z}$; this shows 2°. $\square$

Remark 3.7. (a) Another choice of local coordinates and partition of unity will give another decomposition in (3.1), but this will modify the term of a given order in the localized symbol only by terms of the same and higher order.

(b) When $\nu \notin \mathbb{Z}$ or $\nu < -n+1$, or $\nu \in \mathbb{Z}$ and $\operatorname{tr}_n g$ has a parity that fits with the dimension $n-1$, then $\int ⨍ \operatorname{tr}\operatorname{tr}_n g \, đ\xi' dx'$ will be invariant under such choices, by Remark 3.2.

(c) In the localized situation, we find by integration from Corollary 3.5 that

$$C_0(G, P_{1,\mathrm{D}}) = C_0(\operatorname{tr}_n G, S) + \text{ local terms}, \tag{3.29}$$

for any auxiliary operator S as described there. This reduces the calculation of C_0 on $\mathbb{R}^n_+$ to a calculation of a C_0 on $\mathbb{R}^{n-1}$, modulo local terms (vanishing if $\nu < 1-n$ or $\nu \notin \mathbb{Z}$).

We have hereby obtained (2.7) in the case $A = G$. Now we turn to (2.8), the commutation property of $C_0(G, P_{1,\mathrm{D}})$. We shall give a proof that reduces it to the commutation property for closed manifolds (cf. e.g. [G4]), using a variant of the preceding analysis.

Here we want to consider G given in a different form in the localized situation, namely in the form

$$G = \sum_{j,k\in\mathbb{N}} \Phi_j C_{jk} \Phi_k^* \tag{3.30}$$

explained in Lemma B.6. When $G=\sum_{j,k\in\mathbb{N}}\Phi_j C_{jk}\Phi_k^*$, then

$$\operatorname{Tr}_{\mathbb{R}^n_+}(GQ^N_{\lambda,+})=\operatorname{Tr}_{\mathbb{R}^n_+}(\sum_{j,k\in\mathbb{N}}\Phi_j C_{jk}\Phi_k^* Q^N_{\lambda,+})=\sum_{j,k\in\mathbb{N}}\operatorname{Tr}_{\mathbb{R}^{n-1}}(C_{jk}\Phi_k^* Q^N_{\lambda,+}\Phi_j),$$

since the sum over j,k is rapidly decreasing in the relevant symbol- and operator norms. Here the C_{jk} and $\Phi_k^* Q^N_{\lambda,+}\Phi_j$ are ψdo's on $\mathbb{R}^{n-1}$; to use their properties, we have to investigate $\Phi_k^* Q^N_{\lambda,+}\Phi_j$.

It was a typical feature of the calculations in [GSc] that we used the passage from x'-form to y'-form and vice versa in the compositions in such a way that the case of ψdo symbols over the boundary was reached before it was necessary to verify asymptotic composition formulas such as

$$a(x',\xi',t)\circ b(x',\xi',s)\sim\sum_{\alpha\in\mathbb{N}^{n-1}}\tfrac{1}{\alpha!}D^\alpha_{\xi'}a(x',\xi',t)\partial^\alpha_{x'}b(x',\xi',s), \tag{3.31}$$

with parameters t and s. In the $S^{m,d,s}$-spaces over the boundary, the composition formulas work as usual ([GS1, Th. 1.18]), whereas the mixture of μ-dependence and μ-independence makes it harder to see what goes on in compositions of the ψdbo symbols reaching into the interior of X. Here we rely heavily on the special rational structure of the symbols entering in Q_λ. (The compositions are not covered by [G3].)

In compositions containing $\Phi_k^* Q^N_{\lambda,+}\Phi_j$, we have to deal with terms of the form

$$\operatorname{OPT}(\bar{\hat\varphi}_k(\xi_n,\sigma(\xi')))\operatorname{OP}(q_{-2-J}(x,\xi,\mu))_+\operatorname{OPK}(\hat\varphi_j(\xi_n,\sigma(\xi')))$$

(and their λ-derivatives), which do require asymptotic expansions. In fact, when q_{2-J} is independent of x_n, the two factors to the right give a relatively simple Poisson symbol depending on x', but then in the composition with the trace symbol to the left (depending on ξ' through $\sigma(\xi')$), there will be an infinite expansion as in (3.31), where remainder estimates have to be shown. We shall treat this by going back to the original proof of the pseudodifferential composition rule by Taylor expansion, keeping track of the properties of the terms, in particular the remainder, by use of the exact formulas. A composition to the left with $C_{jk}=\operatorname{OP}'(c_{jk}(x',\xi'))$ does not make the expression harder to deal with, since $\operatorname{OP}'(c_{jk}(x',\xi'))\operatorname{OPT}(\bar{\hat\varphi}_k(\xi_n,\sigma))=\operatorname{OPT}(c_{j,k}(x',\xi')\bar{\hat\varphi}_k(\xi_n,\sigma))$, so we include that in the following.

Proposition 3.8. *Let $C_{jk}=\operatorname{OP}'(c_{jk}(x',\xi'))$ be of order ν and let*

$$S_{J,j,k}=\operatorname{OP}'(c_{jk}(x',\xi'))\operatorname{OPT}(\bar{\hat\varphi}_k(\xi_n,\sigma))\operatorname{OP}(q_{-2-J}(x,\xi,\mu))_+\operatorname{OPK}(\hat\varphi_j(\xi_n,\sigma)). \tag{3.32}$$

For each $J\in\mathbb{N}$, $S_{J,j,k}$ is a ψdo with symbol in $S^{\nu,0,-2-J}$, the $(N-1)$'st λ-derivatives of the symbol lying in $S^{\nu,0,-2N-J}$.

Proof. In view of the expansion in (A.5), $S_{J,j,k}=\sum_{J/2+1\le m\le 2J+1}S_{J,j,k,m}$, where

$$\begin{aligned}S_{J,j,k,m}&=C_{jk}\Phi_k^*\operatorname{OP}\Big(\frac{r_{J,m}(x,\xi)}{(p_{1,2}+\mu^2)^m}\Big)_+\Phi_j\\&=\operatorname{OPT}(c_{jk}(x',\xi')\bar{\hat\varphi}_k(\xi_n,\sigma))\operatorname{OP}\Big(\frac{r_{J,m}(x,\xi)}{(p_{1,2}+\mu^2)^m}\Big)_+\operatorname{OPK}(\hat\varphi_j(\xi_n,\sigma)),\end{aligned} \tag{3.33}$$

and we have to show the property for each of the terms in the expansion. The function $r_{J,m}$ is a polynomial in ξ of degree $2m-J-2\geq 0$. Let us drop the indexations on $r_{J,m}$ and $p_{1,2}$, and simply write the fraction as $r(p+\mu^2)^{-m}=q$.

We first consider the case where these symbols are independent of x_n. Here one has immediately:

$$\mathrm{OP}(q(x',\xi,\mu))_+\,\mathrm{OPK}(\hat\varphi_j(\xi_n,\sigma))=\mathrm{OPK}(b(x',\xi,\mu)),\quad b=h^+_{\xi_n}(q\hat\varphi_j).$$

To prepare for the composition of this Poisson operator with the trace operator $\mathrm{OPT}(c_{jk}\bar{\hat\varphi}_k)$ to the left, we apply the usual procedure for changing an operator from x'-form to y'-form:

$$\begin{aligned}\mathrm{OPK}(b(x',\xi,\mu))v&=\int e^{i(x'-y')\cdot\xi'+ix_n\xi_n}b(x',\xi,\mu)v(y')\,dy'đ\xi\\&=\int e^{i(x'-y')\cdot\xi'+ix_n\xi_n}\sum_{|\alpha|<M}\tfrac{1}{\alpha!}(x'-y')^\alpha\partial_{x'}^\alpha b(y',\xi,\mu)v(y')\,dy'đ\xi\\+\int e^{i(x'-y')\cdot\xi'+ix_n\xi_n}&\sum_{|\alpha|=M}\tfrac{M(x'-y')^\alpha}{\alpha!}\int_0^1(1-h)^{M-1}\partial_{x'}^\alpha b(y'+(x'-y')h,\xi,\mu)v(y')\,dhdy'đ\xi\\&=\sum_{|\alpha|<M}\mathrm{OPK}(b_\alpha(y',\xi,\mu))v+\mathrm{OPK}(b_M(x',y',\xi,\mu))v;\end{aligned}$$

here we have replaced $(x'-y')^\alpha e^{i(x'-y')\cdot\xi'}$ by $D^\alpha_{\xi'}e^{i(x'-y')\cdot\xi'}$ and integrated by parts w.r.t. ξ' (in the oscillatory integrals), and set

$$\begin{aligned}b_\alpha(y',\xi,\mu)&=\tfrac{1}{\alpha!}\overline{D}^\alpha_{\xi'}\partial^\alpha_{y'}b(y',\xi,\mu),\\b_M(x',y',\xi,\mu)&=\sum_{|\alpha|=M}\tfrac{M}{\alpha!}\overline{D}^\alpha_{\xi'}\int_0^1(1-h)^{M-1}\partial^\alpha_{x'}b(y'+(x'-y')h,\xi,\mu)\,dh.\end{aligned}\tag{3.34}$$

Now we compose with $\mathrm{OPT}(c_{jk}\bar{\hat\varphi}_k)$ in front. For each term in the sum over $|\alpha|<M$, this gives:

$$\begin{gathered}\mathrm{OPT}(c_{jk}\bar{\hat\varphi}_k)\,\mathrm{OPK}(b_\alpha)=\mathrm{OP}'(s_\alpha(x',y',\xi',\mu)),\text{ where}\\s_\alpha(x',y',\xi',\mu)=\int c_{jk}\bar{\hat\varphi}_k\tfrac{1}{\alpha!}\overline{D}^\alpha_{\xi'}\partial^\alpha_{y'}h^+_{\xi_n}(q\hat\varphi_j)\,đ\xi_n=\int c_{jk}\bar{\hat\varphi}_k\tfrac{1}{\alpha!}\overline{D}^\alpha_{\xi'}\partial^\alpha_{y'}(q(y',\xi,\mu)\hat\varphi_j)\,đ\xi_n.\end{gathered}$$

In the last step it is used that $\overline{D}^\alpha_{\xi'}\partial^\alpha_{y'}$ commutes with $h^+_{\xi_n}$ and that $\overline{D}^\alpha_{\xi'}\partial^\alpha_{y'}(q\hat\varphi_j)$ is a sum of terms with a similar structure as $q\hat\varphi_j$, so that one can remove h^+ in the calculation of the plus-integral, as in (B.5). More precisely,

$$\begin{gathered}\overline{D}^\alpha_{\xi'}\partial^\alpha_{y'}(q\hat\varphi_j)=\sum_{\gamma\leq\alpha}\tbinom{\alpha}{\gamma}\overline{D}^{\alpha-\gamma}_{\xi'}\partial^\alpha_{y'}q\overline{D}^\gamma_{\xi'}\hat\varphi_j,\text{ where}\\\overline{D}^{\alpha-\gamma}_{\xi'}\partial^\alpha_{y'}q=\sum_{m\leq m'\leq m+|\alpha-\gamma|}\frac{r_{m'}(y',\xi)}{(p+\mu^2)^{m'}},\quad\overline{D}^\gamma_{\xi'}\hat\varphi_j=\sum_{|j'-j|\leq|\gamma|,j'\geq0}r'_{j'}(\xi')\hat\varphi_{j'};\end{gathered}\tag{3.35}$$

here the $r_{m'}(y',\xi)$ are polynomials in ξ of degree $\le 2m'-J-2-|\alpha-\gamma|$ (vanishing if this number is <0) and the $r'_{j'}(\xi')$ are in $S^{-|\gamma|}$ (found from (B.2)). It follows that

$$\overline{D}^{\alpha}_{\xi'}\partial^{\alpha}_{y'}(q(y',\xi,\mu)\hat\varphi_j(\xi_n,\sigma))=\sum\frac{r_{\alpha,m',j'}(y',\xi)}{(p+\mu^2)^{m'}}\hat\varphi_{j'},\tag{3.36}$$

where the sum is over $m\le m'\le m+|\alpha|$, $|j-j'|\le|\alpha|$, $j'\ge 0$, and the $r_{\alpha,m',j'}(y',\xi)$ are in $S^{2m'-J-2-|\alpha|}(\mathbb{R}^{n-1},\mathbb{R}^n)$; they are polynomials in ξ_n of degree $\le 2m'-J-2$, vanishing if this number is <0. Each $r_{\alpha,m',j'}(p+\mu^2)^{-m'}$ is decomposed in simple fractions with poles $\pm i\kappa^{\pm}$ as in (A.7):

$$\frac{r_{\alpha,m',j'}(y',\xi)}{(p(y',\xi)+\mu^2)^{m'}}=\sum_{1\le m''\le m'}\frac{r^{+}_{\alpha,m',j',m''}(y',\xi',\mu)}{(\kappa^{+}(y',\xi',\mu)+i\xi_n)^{m''}}+\sum_{1\le m''\le m'}\frac{r^{-}_{\alpha,m',j',m''}(y',\xi',\mu)}{(\kappa^{-}(y',\xi',\mu)-i\xi_n)^{m''}};\tag{3.37}$$

here the numerators are of order $m''-|\alpha|-J-2$, lying in $S^{-|\alpha|,0,m''-J-2}$ (since the μ-independent factors coming from derivatives of $\hat\varphi_j$ have order down to $-|\alpha|$), and each λ-differentiation lowers the third upper index by 2.

We can now apply Lemma B.4 to the compositions of $c_{jk}\bar{\hat\varphi}_k$ with (3.37) (recalling from (B.1) that the normalized Laguerre functions have a factor $(2\sigma)^{\frac12}$). This gives ψdo symbols in (x',y')-form on $\mathbb{R}^{n-1}$, lying in $S^{\nu-|\alpha|,0,m''-J-2}\cdot S^{1,0,-m''-1}\subset S^{\nu+1-|\alpha|,0,-J-3}$ if $j'\ne k$, and in $S^{\nu-|\alpha|,0,-J-2}$ if $j'=k$.

Summing over the indices we find that $s_\alpha(y',\xi',\mu)\in S^{\nu-|\alpha|,0,-J-2}$, and each λ-differentiation lowers the third upper index by 2.

From the sequence of symbols s_α, one constructs a symbol $s\in S^{\nu,0,-J-2}$ having the asymptotic expansion $s\sim\sum_{\alpha\in\mathbb{N}^{n-1}}s_\alpha$ in the usual way. There remains the question of how well this represents a symbol of the operator given in (3.33). This will be dealt with by a consideration of the remainder after M terms, using the formula in (3.34).

We shall use the strategy explained after (3.7). In order to be able to distinguish between the various covariables, we shall write here explicitly $\hat\varphi(\xi_n,[\xi'])$ and $\hat\varphi(\xi_n,[\eta'])$ instead of $\hat\varphi(\xi_n,\sigma)$. Consider, for large M,

$$\begin{aligned}&\mathrm{OPT}(c_{jk}\bar{\hat\varphi}_k)\,\mathrm{OPK}(b_M)v(x')\\&=\int_{\mathbb{R}^{4n-3}}e^{i(x'-z')\cdot\xi'+i(z'-y')\cdot\eta'}c_{jk}(x',\xi')\bar{\hat\varphi}_k(\xi_n,[\xi'])h^{+}_{\xi_n}\Big[\sum_{|\alpha|=M}\tfrac{M}{\alpha!}\overline{D}^{\alpha}_{\eta'}\\&\quad\int_0^1(1-h)^{M-1}\partial^{\alpha}_{z'}q(y'+(z'-y')h,\eta',\xi_n,\mu)\hat\varphi_j(\xi_n,[\eta'])\Big]v(y')\,dh đ\xi_n dy' đ\eta' dz' đ\xi'\\&=\int_{\mathbb{R}^{4n-3}}e^{i(x'-z')\cdot\xi'+i(z'-y')\cdot\eta'}c_{jk}\bar{\hat\varphi}_k(\xi_n,[\xi'])\sum_{|\alpha|=M}\tfrac{M}{\alpha!}\overline{D}^{\alpha}_{\eta'}\int_0^1(1-h)^{M-1}\\&\qquad\times\partial^{\alpha}_{z'}q(y'+(z'-y')h,\eta',\xi_n,\mu)\hat\varphi_j(\xi_n,[\eta'])v(y')\,dh đ\xi_n dy' đ\eta' dz' đ\xi';\end{aligned}\tag{3.38}$$

here we have again commuted $\overline{D}^{\alpha}_{\eta'}\partial^{\alpha}_{z'}$ and $h^{+}_{\xi_n}$, and removed the latter because of the

special structure as rational functions. The kernel of this operator is

$$K_M(x',y',\mu)=\int_{\mathbb{R}^{3n-2}} e^{i(x'-z')\cdot\xi'+i(z'-y')\cdot\eta'}c_{jk}\bar{\hat\varphi}_k(\xi_n,[\xi'])\sum_{|\alpha|=M}\tfrac{M}{\alpha!}\overline{D}^{\alpha}_{\eta'}$$
$$\times\int_0^1(1-h)^{M-1}\partial^\alpha_{z'}q(y'+(z'-y')h,\eta',\xi_n,\mu)\hat\varphi_j(\xi_n,[\eta'])\,dh đ\xi_n d\eta' dz' đ\xi'. \tag{3.39}$$

Performing the differentiations $\overline{D}^\alpha_{\eta'}\partial^\alpha_{z'}$, one gets a sum of similar expressions, now with order M steps lower in the Poisson operator. Consider one term in such an expression, it is of the form (3.39) with q replaced by a fraction $r'(p+\mu^2)^{-m'}$ as in (3.36), with r' in $S^{2m'-J-2-M}$:

$$\begin{aligned}&\mathrm{OPT}(c_{jk}\bar{\hat\varphi}_k)\,\mathrm{OPK}(b')v(x')\\&=\int_{\mathbb{R}^{4n-3}} e^{i(x'-z')\cdot\xi'+i(z'-y')\cdot\eta'}c_{jk}\bar{\hat\varphi}_k(\xi_n,[\xi'])\int_0^1(1-h)^{M-1}\\&\times\frac{r'(y'+(z'-y')h,\eta',\xi_n)}{(p(y'+(z'-y')h,\eta',\xi_n)+\mu^2)^{m'}}\hat\varphi_j(\xi_n,[\eta'])v(y')\,dh đ\xi_n dy' đ\eta' dz' đ\xi'.\end{aligned}\tag{3.40}$$

Insert in (3.40) the expansion of the denominator from Lemma A.1:

$$(p+\mu^2)^{-m'}=\sum_{0\le i<L}c_{m',i}\mu^{-2m'-2i}p^i+\mu^{-2m'-2L}p'_L. \tag{3.41}$$

The composed operators resulting from the terms in the sum over i are of the form $\mu^{-2m'-2i}S_i$, where S_i is a μ-independent ψdo on $\mathbb{R}^{n-1}$ of order $\le\nu+2m'+2i-J-2-M$ (with $i<L$). These operators belong to the calculus. Taking M large in comparison with $2L$ we can assure that their kernels have any given finite degree of smoothness.

The contribution from $\mu^{-2m'-2L}p'_L$ equals $\mu^{-2m'-2L}S'$, where S' has a form as in (3.40) with $(p+\mu^2)^{-m'}$ replaced by p'_L; it is a ψdo on $\mathbb{R}^{n-1}$ of order $\le\nu+2m'+2L-J-2-M$. The estimates (A.2) imply that the entries satisfy all symbol estimates *with uniform estimates in* $\mu\in\Gamma$ for $|\mu|\ge1$. Then S' maps

$$S'\colon H^s(\mathbb{R}^{n-1})\to H^{s-\nu-2m'-2L+J+2+M}(\mathbb{R}^{n-1}),\ \text{any } s\in\mathbb{R},$$

with uniform estimates in $\mu\in\Gamma$ for $|\mu|\ge1$, since the norms of such mappings are estimated in terms of the symbol seminorms. Taking M large in comparison with $2L$, we can achieve that the kernel of S' is a smooth as we want, with uniform bounds on the (x',y')-derivatives.

It follows that

$$S_{J,j,k,m}-\mathrm{OP}'\big(s(x',y',\xi',\mu)\big)=S_{J,j,k,m}-\mathrm{OP}'\big(\sum_{|\alpha|<M}s_\alpha\big)-\mathrm{OP}'\big(s-\sum_{|\alpha|<M}s_\alpha\big) \tag{3.42}$$

has a kernel expansion as in (3.7) (for $d=-J-2$), where L and the degree of smoothness can be as high as we want, since M can be taken arbitrarily large. Since differentiation in λ lowers the d-index by two steps, and $\partial_\lambda=-2\mu\partial_\mu$, the information required for (3.8) is

likewise found in this way. This shows that $S_{J,j,k,m}$ is a ψdo with symbol $\sim s$ in $S^{\nu,0,-J-2}$. Its λ-derivatives are treated in a similar way.

This shows the proposition in the case where the symbols are independent of x_n. Now let q_{-2-J} depend on x_n, and consider again (3.32). By a Taylor expansion in x_n, q_{-2-J} is of the form

$$q_{-2-J}(x,\xi,\mu)=\sum_{0\le r<M}\tfrac{1}{r!}x_n^r\partial_{x_n}^r q_{-2-J}(x',0,\xi,\mu)\\ +\tfrac{1}{(M-1)!}x_n^M\int_0^1(1-h)^{M-1}\partial_{x_n}^M q_{-2-J}(x',hx_n,\xi,\mu)\,dh. \tag{3.43}$$

For the terms in the sum over r, we observe that the factor x_n^r goes together with the trace operator to the left $\mathrm{OPT}(c_{jk}(x',\xi')\bar{\hat\varphi}_k(\xi_n,\sigma))$ and replaces its symbol by $c_{jk}D_{\xi_n}^r\bar{\hat\varphi}_k(\xi_n,\sigma)$ [G2, (2.4.14)]. This lowers the degree — but not the d-index — by r steps; more precisely it gives $c_{jk}[\xi']^{-r}$ times a linear combination of adjacent Laguerre functions, cf. (B.2). The resulting composition is of the type we have dealt with above, giving a symbol in $S^{\nu-r,0,-2-J}$.

The last term in (3.43) is a linear combination of terms of the form

$$x_n^M\int_0^1(1-h)^{M-1}\partial_{x_n}^M\Big(\frac{r(x',hx_n,\xi)}{(p(x',hx_n,\xi)+\mu^2)^m}\Big)\,dh,$$

with r polynomial in ξ of order $2m-2-J\ge 0$; consider one such term. Insert the expansion (3.41) of $(p+\mu^2)^{-m}$. The composed operators resulting from the terms in the sum over i are of the form $\mu^{-2m-2i}S_i$ with S_i of order $\le \nu+2m+2i-2-J-M$ (with $i<L$). The operators belong to the calculus, with more smoothness of the kernels, the larger M is taken in comparison with $2L$. Finally, the contribution from p'_L is of the form $\mu^{-2m-2L}\widetilde S$, with

$$\begin{aligned}\widetilde S&=\mathrm{OPT}(c_{jk}D_{\xi_n}^M\bar{\hat\varphi}_k(\xi_n,\sigma))\,\mathrm{OP}(\tilde q_L(x,\xi,\mu))_+\,\mathrm{OPK}(\hat\varphi_j(\xi_n,\sigma)),\\ \tilde q_L(x,\xi,\mu)&=\tfrac{1}{(M-1)!}\int_0^1(1-h)^{M-1}\partial_{x_n}^M\big(r(x',hx_n,\xi)p'_L(x',hx_n,\xi,\mu)\big)\,dh.\end{aligned} \tag{3.44}$$

The estimates (A.2) imply that for all indices:

$$|D_x^\beta D_\xi^\alpha \tilde q_L(x,\xi,\mu)|\le C'_{\alpha,\beta}\langle\xi\rangle^{2m-2-J+2L-|\alpha|}, \tag{3.45}$$

with $C'_{\alpha,\beta}$ independent of $\mu\in\Gamma$ for $|\mu|\ge 1$. Then the resulting operator $\widetilde S$ maps $H^s(\mathbb{R}^{n-1})$ to $H^{s-\nu-2m+2+J-2L+M+1}(\mathbb{R}^{n-1})$ for any $s\in\mathbb{R}$, with uniform estimates in $\mu\in\Gamma$ for $|\mu|\ge 1$. Taking L large, and M large in comparison with $2L$, we can obtain that the resulting kernel is a smooth as we want, with coefficient μ^{-2m-2L} of as low order as we want.

As in the preceding considerations, this allows us to conclude that the whole composed operator belongs to the calculus, with symbol properties as asserted. $\square$

Remark 3.9. In the above proof, one can moreover keep track of the size of symbol seminorms in their dependence on j and k. It follows from Lemma B.4 and the results on ξ'-derivatives and ξ_n-derivatives we used, that for each J, the symbol seminorms resulting from calculations with Laguerre funtions are only polynomially increasing in j and k. Thus if the symbol seminorms of the c_{jk} are rapidly decreasing in j and k, as they are when the c_{jk} come from an s.g.o. as in Lemma B.6, summations in j and k of operators $S_{J,j,k}$ will converge in the relevant symbol seminorms.

Note that for $J \geq 1$, the $(N-1)$'st derivatives of the symbols of the operators $S_{J,j,k}$ lie in $S^{\nu,0,-2N-1} \subset S^{\nu-2N-1,0,0} \cap S^{\nu,-2N-1,0}$ (cf. (3.3)); hence in their diagonal kernel expansions as in (3.18), the sum over k starts with $k \geq 1$, so that they contribute only locally to the coefficient of μ^{-2N}. Thus only the terms with $J = 0$,

$$\begin{aligned}\frac{\partial_\lambda^{N-1}}{(N-1)!} S_{0,j,k} &= \frac{\partial_\lambda^{N-1}}{(N-1)!} C_{jk} \Phi_k^* \operatorname{OP}((p_{1,2}+\mu^2)^{-1})_+ \Phi_j \\ &= C_{jk} \Phi_k^* \operatorname{OP}((p_{1,2}+\mu^2)^{-N})_+ \Phi_j\end{aligned} \tag{3.46}$$

can contribute nonlocally to the coefficient of μ^{-2N}. We can use the analysis in the proof of Proposition 3.8 to distinguish the contributing part still further:

Proposition 3.10. *We have for each $N \geq 1$ that*

$$C_{jk} \Phi_k^* \operatorname{OP}((p_{1,2}+\mu^2)^{-N})_+ \Phi_j = \operatorname{OP}'\big(\delta_{jk} c_{jk}(x',\xi') \alpha^{(N)}(y',\xi',\mu)\big) + \widetilde{S}_{0,j,k,N}, \tag{3.47}$$

where $\alpha^{(N)}$ is as defined in (A.12) *and $\widetilde{S}_{0,j,k,N}$ has symbol in $S^{\nu+1,0,-2N-1}$.*

Proof. First let q be independent of x_n, and consider how $q_{-2} = (p+\mu^2)^{-1}$ (with $p = p_{1,2}$) enters in the proof of Proposition 3.8, for $N = 1$. Whenever we apply a differentiation $\partial_{x'}^\alpha$ with $|\alpha| > 0$ to q, it produces a sum of rational functions with denominators $(p+\mu^2)^{-m}$ with $m \geq 2$ and μ-independent numerators, so only the undifferentiated term q_{-2} retains the power -1. Therefore the resulting terms in (3.35) with $|\alpha| > 0$ have denominators $(p+\mu^2)^{-m'}$ with $m' \geq 2$. Then when we decompose in simple fractions and apply Lemma B.4, we get symbols lying in $S^{\nu+1,0,-3}$. There remains the term

$$c_{jk}(x',\xi') \int \bar{\hat{\varphi}}_k(\xi_n,\sigma)(p(y',\xi',\xi_n)+\mu^2)^{-1} \hat{\varphi}_j(\xi_n,\sigma)\, d\!\!{}^-\xi_n.$$

Here Lemma B.4 shows that it gives a symbol in $S^{\nu+1,0,-3}$ when $j \neq k$, whereas it gives $c_{jk}(x',\xi')\alpha^{(1)}(y',\xi',\mu) \in S^{\nu,0,-2}$ when $j = k$ (note that the norming factors $(2\sigma)^{\frac{1}{2}}$ in the Laguerre functions eliminate the division by 2σ in (B.8)).

This shows the claim for $N = 1$, and in view of the information on λ-derivatives, it follows for general N.

When q depends on x_n, we consider the Taylor expansion (3.43). Again, any derivative $\partial_{x_n}^r$ with $r > 0$ produces rational functions with $p+\mu^2$ in powers ≥ 2 in the denominators. So we find that the Taylor terms, except for the first one, lead to symbols with lower third index. The first term is dealt with above. $\square$

Theorem 3.11. *Consider a localized situation, where G is given in the form* (3.30) *and the symbols $c_{jk}(x',\xi')$ have compact x'-support. Define the diagonal sum*

$$C=\sum_{j\in\mathbb{N}}C_{jj},\ \text{ with symbol } c(x',\xi')=\sum_{j\in\mathbb{N}}c_{jj}(x',\xi'). \tag{3.48}$$

Then

$$\operatorname{Tr}_{\mathbb{R}^n_+}(GQ^N_{\lambda,+})=\operatorname{Tr}_{\mathbb{R}^{n-1}}\operatorname{OP}'(c(x',\xi')\alpha^{(N)}(x',\xi',\mu))+\operatorname{Tr}_{\mathbb{R}^{n-1}}\widetilde{S}', \tag{3.49}$$

where $\widetilde{S}'$ has symbol in $S^{\nu+1,0,-2N-1}$. It follows that for $N>(\nu+n-1)/2$, the trace has an expansion (with all $c'_k=0$ if $\nu\notin\mathbb{Z}$):

$$\operatorname{Tr}_{\mathbb{R}^n_+}(GQ^N_{\lambda,+})\sim\sum_{j\geq 1}c_j(-\lambda)^{\frac{n+\nu-j}{2}-N}+\sum_{k\geq 0}(c'_k\log(-\lambda)+c''_k)(-\lambda)^{-\frac{k}{2}-N}, \tag{3.50}$$

where

$$c''_0=\int\!\!⨍\operatorname{tr}c(x',\xi')\,đ\xi'dx'+\ \text{local terms} \tag{3.51}$$

(with vanishing local terms if $\nu\in\mathbb{R}\setminus\mathbb{Z}$ or $\nu<1-n$).

Proof. For each j,k, we conclude from Proposition 3.10 that

$$\begin{aligned}\operatorname{Tr}_{\mathbb{R}^n_+}(\Phi_jC_{jk}\Phi_k^*Q^N_{\lambda,+})&=\operatorname{Tr}_{\mathbb{R}^{n-1}}(C_{jk}\Phi_k^*Q^N_{\lambda,+}\Phi_j)\\&=\operatorname{Tr}_{\mathbb{R}^{n-1}}\operatorname{OP}'(\delta_{jk}c_{jk}(x',\xi')\alpha^{(N)}(y',\xi',\mu))+\operatorname{Tr}_{\mathbb{R}^{n-1}}\widetilde{S}'_{0,j,k,N}\\&=\operatorname{Tr}_{\mathbb{R}^{n-1}}\operatorname{OP}'(\delta_{jk}c_{jk}(x',\xi')\alpha^{(N)}(x',\xi',\mu))+\operatorname{Tr}_{\mathbb{R}^{n-1}}\widetilde{S}'_{0,j,k,N}.\end{aligned}$$

with pseudodifferential operators $\tilde{S}'_{0,j,k,N}$ having symbols in $S^{\nu+1,0,-2N-1}$. For the last equality it is used that the trace is calulated from the diagonal values of the kernel, where y' is taken equal to x'. The formula (3.49) follows by summation over j and $k\in\mathbb{N}$, using that $(C_{jk})_{j,k\in\mathbb{N}}$ is rapidly decreasing in j and k and that the symbol seminorms of the composed operators are polynomially controlled in j and k, as indicated in Remark 3.9.

Since $c_{jj}\alpha^{(N)}\in S^{\nu-2N,0,0}\cap S^{\nu,-2N,0}$, the trace expansion follows from (3.5) with the statement (3.51) as in the proof of Theorem 3.3. (We have replaced j by $j+1$ in (3.18) and inserted $\mu=(-\lambda)^{\frac12}$, to facilitate comparison with (1.10).) $\square$

We have as an immediate corollary (using Proposition B.3 for $GG^{(N)}_\lambda$):

Theorem 3.12. *Let G be a singular Green operator on X of order $\nu\in\mathbb{R}$ and class* 0*. Consider a localization $\underline{G}$ to $\overline{\mathbb{R}}^n_+$ as described after Lemma* 3.1*, write*

$$\underline{G}=\sum_{j,k\in\mathbb{N}}\Phi_j\underline{C}_{jk}\Phi_k^*,\quad \underline{C}=\sum_{j\in\mathbb{N}}\underline{C}_{jj}, \tag{3.52}$$

with $\underline{C} = \mathrm{OP}'(c(x',\xi'))$*, and carry* $\underline{C}$ *back to an operator* C *on* X' *by use of the considered coordinate mappings. Then*

$$C_0(G, P_{1,\mathrm{D}}) = \int ⨍ \operatorname{tr} c(x',\xi')\, đ\xi' dx' + \text{ local terms,} \tag{3.53}$$

with vanishing local terms if $\nu < -n+1$ *or* $\nu \in \mathbb{R}\setminus\mathbb{Z}$*. In particular, we have for the localized operators as well as for the operators on* X, X'*:*

$$\begin{aligned} C_0(\underline{G}, \underline{P}_{1,\mathrm{D}}) &= C_0(\underline{C}, \underline{S}) + \text{ local terms,}\\ C_0(G, P_{1,\mathrm{D}}) &= C_0(C, S) + \text{ local terms,} \end{aligned} \tag{3.54}$$

for any auxiliary elliptic second-order operator S *on* X' *with no principal symbol eigenvalues on* $\mathbb{R}_-$*.*

This can be used to show the commutation property for $C_0(G, P_{1,\mathrm{D}})$:

Theorem 3.13. *When* G *and* G' *are singular Green operators of orders* ν *resp.* ν' *and class 0, then* $C_0([G,G'], P_{1,\mathrm{D}})$ *is locally determined; it vanishes if* $\nu+\nu' \notin \mathbb{Z}$ *or* $\nu+\nu' < 1-n$*.*

Proof. Expand the localized versions in series with Laguerre operators,

$$\underline{G} = \sum_{j,k\in\mathbb{N}} \Phi_j \underline{C}_{jk} \Phi_k^*, \quad \underline{G'} = \sum_{j,k\in\mathbb{N}} \Phi_j \underline{C}'_{jk} \Phi_k^*.$$

In view of (B.13),

$$\begin{aligned} [\underline{G}, \underline{G'}] &= \sum_{j,k,l,m} \Phi_j \underline{C}_{jk} \Phi_k^* \Phi_l \underline{C}'_{lm} \Phi_m^* - \sum_{j,k,l,m} \Phi_j \underline{C}'_{jk} \Phi_k^* \Phi_l \underline{C}_{lm} \Phi_m^*\\ &= \sum_{j,k,m} \Phi_j (\underline{C}_{jk}\underline{C}'_{km} - \underline{C}'_{jk}\underline{C}_{km}) \Phi_m^*, \end{aligned}$$

and the diagonal sum associated with this operator as in (3.52) is:

$$\sum_{j,k} (\underline{C}_{jk}\underline{C}'_{kj} - \underline{C}'_{jk}\underline{C}_{kj}) = \sum_{j,k} \underline{C}_{jk}\underline{C}'_{kj} - \sum_{j,k} \underline{C}'_{kj}\underline{C}_{jk} = \sum_{j,k} [\underline{C}_{jk}, \underline{C}'_{kj}].$$

This series converges in the relevant symbol seminorms, in view of the rapid decrease of the $\underline{C}_{jk}$ and $\underline{C}'_{jk}$ for $j,k\to\infty$. Then it follows from Theorem 3.12 that

$$C_0([\underline{G}, \underline{G'}], \underline{P}_{1,\mathrm{D}}) = C_0(\sum_{j,k} [\underline{C}_{jk}, \underline{C}'_{kj}], \underline{S}) + \text{ local terms .}$$

Now we use the fact known for closed manifolds (cf. e.g. [G4]) that each $C_0([\underline{C}_{jk}, \underline{C}'_{kj}], \underline{S})$ is local, defined from specific homogeneous terms in the symbols; then so is the sum. In particular, they vanish in the cases $\nu+\nu' \notin \mathbb{Z}$ or $\nu+\nu' < 1-n$; then so does the sum. The resulting statements carry back to the manifold situation. Hence $C_0([G,G'], P_{1,\mathrm{D}})$ is local. $\square$

It may be remarked that when $G = \sum_{j,k\in\mathbb{N}} \Phi_j C_{jk} \Phi_k^*$ on $\overline{\mathbb{R}}^n_+$ with $C_{jk} = \mathrm{OP}'(c_{jk}(x',\xi'))$, then the symbol of G has the same principal part as $\sum_{j,k\in\mathbb{N}} \hat{\varphi}_j(\xi_n,\sigma) c_{jk}(x',\xi') \bar{\hat{\varphi}}_k(\eta_n,\sigma)$, but need not equal the full symbol. Likewise, the normal trace $\operatorname{tr}_n G$ has the same principal symbol as the diagonal sum $\sum_{j\in\mathbb{N}} C_{jj}$, but not always the same full symbol. In any case, the above considerations show that they give the same contribution to $C_0(G, P_{1,\mathrm{D}})$ (modulo local terms, if $\nu \in \mathbb{Z}$, $\nu \geq -n$).

This ends the proof that $C_0(G, P_{1,\mathrm{D}})$ is a quasi-trace on s.g.o.s. We can also describe cases where it is a trace; more about that below.

The local terms, we have talked about so far, are defined from the symbols on X. But in fact, it is only the behavior in an arbitrarily small neighborhood of X' that enters:

Assume that a normal coordinate x_n has been chosen such that a suitable neighborhood X_1 of X' in X is represented by the product $X_1 = X' \times [0,1]$, with $x \in X_1$ written as $x = (x', x_n)$, $x' \in X'$ and $x_n \in [0,1]$. We can also assume that $E|_{X_1}$ is the lifting of $E|_{X'}$. Let $\chi(t)$ be a C^∞ function on $\overline{\mathbb{R}}_+$ that is 1 for $t \leq 1$ and 0 for $t \geq 2$, and, for $\varepsilon < \frac{1}{2}$, let

$$\chi_\varepsilon(x) = \begin{cases} \chi(x_n/\varepsilon) \text{ for } x \in X_1, \\ 0 \text{ for } x \in X \setminus X_1. \end{cases} \tag{3.55}$$

Then we can write

$$G = G_{\mathrm{b}} + G_{\mathrm{i}}; \quad G_{\mathrm{b}} = \chi_{\varepsilon/4} G \chi_{\varepsilon/4}, \tag{3.56}$$

here G_{b} is supported in $X_{\varepsilon/2} = X' \times [0, \varepsilon/2]$, and G_{i} is of order $-\infty$ (in particular it is trace-class). The auxiliary local coordinates can be chosen such that the variable x_n is preserved for the patches intersecting with X_1; then $\operatorname{tr}_n G_{\mathrm{b}}$ has a meaning as a ψdo on X'.

Corollary 3.14.

(i) *With the decomposition defined above,*

$$C_0(G, P_{1,\mathrm{D}}) = C_0(G_{\mathrm{b}}, P_{1,D}) + \operatorname{Tr}_X G_{\mathrm{i}} + \textit{ local terms}, \tag{3.57}$$

where the local terms depend on the symbol of P_1 and the first $\nu + n$ symbols of G on X_ε only.

(ii) *If $\nu \in \mathbb{R} \setminus \mathbb{Z}$ or $\nu < -n+1$, then*

$$C_0(G, P_{1,\mathrm{D}}) = \mathrm{TR}_{X'}(\operatorname{tr}_n G_{\mathrm{b}}) + \operatorname{Tr}_X G_{\mathrm{i}}. \tag{3.58}$$

For such values of ν, $C_0(G, P_{1,\mathrm{D}})$ is a canonical trace on the operators G:

$$C_0(G, P_{1,\mathrm{D}}) = \mathrm{TR}\, G, \tag{3.59}$$

in the sense that it satisfies:

(1) $C_0(G, P_{1,\mathrm{D}})$ *is independent of P_1 if $\nu \notin \mathbb{Z}$ or $\nu < 1-n$;*
(2) $C_0([G, G'], P_{1,\mathrm{D}}) = 0$ *if $\nu + \nu' \notin \mathbb{Z}$ or $\nu + \nu' < 1 - n$.*

(iii) *If* $\nu < 1-n$,

$$\operatorname{TR} G = \operatorname{Tr}_X G. \tag{3.60}$$

Proof. For (i), note that since $\chi_{\varepsilon/4} = \chi_{\varepsilon/2}\chi_{\varepsilon/4}$

$$\operatorname{Tr}(G_{\rm b} R^N_\lambda) = \operatorname{Tr}(\chi_{\varepsilon/2}\chi_{\varepsilon/4} G \chi_{\varepsilon/2} R^N_\lambda) = \operatorname{Tr}(\chi_{\varepsilon/4} G \chi_{\varepsilon/2} R^N_\lambda \chi_{\varepsilon/2}),$$

which shows that only the behavior of R_λ on X_ε and the behavior of G on $X_{\varepsilon/2}$ enter in the calculation of $C_0(G_{\rm b}, P_{1,{\rm D}})$.

As noted in the preceding theorems, the formulas are exact without unspecified local terms, when ν is as in (ii). Moreover, $C_0(G_{\rm b}, P_{1,{\rm D}}) = C_0(\operatorname{tr}_n G_{\rm b}, S)$ then by (3.29), for the allowed auxiliary elliptic operators S. For the manifold X' without boundary we have from [KV], [L], [G4] that $C_0(\operatorname{tr}_n G_{\rm b}, S) = \operatorname{TR}(\operatorname{tr}_n G_{\rm b})$ for such ν. This shows (3.58). The statements in (1) and (2) have been shown further above.

For (iii), we note that when $\nu < 1-n$, $G_{\rm b}$ is trace-class with continuous kernel, and

$$\operatorname{Tr} G_{\rm b} = \operatorname{Tr}_{X_\varepsilon} G_{\rm b} = \int_{X'} \int_0^\varepsilon K(G_{\rm b}, x, x)\, dx_n dx' = \operatorname{Tr}_{X'}(\operatorname{tr}_n G_{\rm b}). \tag{3.61}$$

Then $C_0(G, P_{1,{\rm D}}) = \operatorname{Tr} G_{\rm b} + \operatorname{Tr} G_{\rm i} = \operatorname{Tr} G$. $\square$

If $G_{\rm b}$ is written in the local coordinates in the composition form (3.52), one can show similar formulas with C instead of $\operatorname{tr}_n G_{\rm b}$.

When ν is integer, it may be so that $\operatorname{tr}_n G_{\rm b}$ or C has a parity that fits with the dimension $n-1$, cf. (1.4)–(1.5)ff. One could then try to show a canonical trace property of $C_0(G, P_{1,{\rm D}})$ as in the corresponding situation for operators given directly on the closed manifold X' as in [G4]. This is further supported by the fact that since $\kappa^+(x', -\xi', \mu) = \kappa^-(x', \xi', \mu)$, $\alpha^{(1)}$ is even-even, cf. (A.12). However, there are a lot of terms that give local contributions; not only lower order terms in compositions but notably all the $c_{lm}(x', \xi')$ with $l \neq m$ in the symbol $\sum_{l,m\in\mathbb{N}} c_{lm}\hat\varphi_l \bar{\hat\varphi}_m$ of G; cf. Lemma B.4: For $l > m$, the resulting symbols involve κ^+, for $l < m$ κ^-, on the principal level. A high degree of symmetry seems required, or we can restrict the auxiliary operators to a nicer type:

Consider G as defined on the cylinder $X' \times \overline{\mathbb{R}}_+$ (after being cut down to a neighborhood X_ε of X'). Referring to this fixed choice of normal coordinate, we assume that the symbol of P_1 is not only even-even with respect to ξ (as any differential operator is), but it is so with respect to ξ'. This holds when there are no terms with D_{x_n} times a first-order operator in x', for example when

$$P_1 = D^2_{x_n} + P'_1, \tag{3.62}$$

where P'_1 is a second-order elliptic differential operator on X'. We can take P'_1 with positive principal symbol, to assure strong ellipticity of P_1 (a little more generality could be allowed).

Assume that $G = \operatorname{OPG}(\sum_{j,k\in\mathbb{N}} c_{jk}\hat\varphi_j \bar{\hat\varphi}_k)$ (expressed in local coordinates on X', using x_n as the normal coordinate), with the $c_{jk}(x', \xi')$ all being even-even or all being even-odd. Then when $GQ^N_{\lambda,+}$ and $GG^{(N)}_\lambda$ are calculated, we find that

$$\operatorname{tr}_n GQ^N_{\lambda,+} \text{ and } \operatorname{tr}_n GG^{(N)}_\lambda$$

are even-even of order $\nu - 2N$, resp. even-odd of order $\nu - 2N$. An application of the proof of [GS1, Th. 2.1] to these two ψdo's on X' gives that in the even-even case, the local contributions to the coefficient of μ^0 and to $\log \mu$ vanish if $n-1$ is odd, and in the even-odd case, the local contributions to the coefficient of μ^0 and to $\log \mu$ vanish if $n-1$ is even. So indeed, restricting the auxiliary operators to those of the form (3.62), we have a canonical trace in suitable parity cases. We have shown:

Theorem 3.15. *Let G be given on the cylinder $X' \times \overline{\mathbb{R}}_+$ with points (x', x_n) and let P_1, in addition to being strongly elliptic with scalar principal symbol, be of the form* (3.62). *Let G be of order $\nu \in \mathbb{Z}$, with symbol $\sum_{j,k\in\mathbb{N}} c_{jk}(x',\xi')\hat{\varphi}_j(\xi_n,\sigma)\bar{\hat{\varphi}}_k(\eta_n,\sigma)$ (for a choice of local coordinates on X', with x_n as normal coordinate). Then*

$$C_0(G, P_{1,\mathrm{D}}) = \mathrm{TR}(\mathrm{tr}_n\, G) \tag{3.63}$$

holds in the following cases:

(i) *All the c_{jk} are even-even and n is even.*

(ii) *All the c_{jk} are even-odd and n is odd.*

In this sense, we may say that s.g.o.s have the canonical trace

$$\mathrm{TR}\, G = \mathrm{TR}_{X'}(\mathrm{tr}_n\, G), \tag{3.64}$$

in the above cases (i) and (ii). One can also show that $C_0([G, G'], P_{1,\mathrm{D}}) = 0$ when, in the case n even, the c_{jk} and c'_{jk} are either all even-even or all even-odd; in the case n odd it holds when all the c_{jk} are even-even and all the c'_{jk} are even-odd.

If G is given in the form (3.52), the statements hold with $\mathrm{tr}_n\, G$ replaced by $C = \sum_{j\in\mathbb{N}} C_{jj}$.

One could also introduce a parity concept directly for s.g.o. symbols g, but it involves some unpractical shifts: Recall that in the polyhomogeneous expansion $g \sim \sum_{l\geq 0} g_{\nu-1-l}$, the homogeneity degree of the l'th term is $\nu - 1 - l$, one step lower than the order. Since the φ_j depend on ξ' through $\sigma(\xi') = [\xi']$, the parity statements on the c_{jk} correspond to the opposite parity in ξ' for g, relative to the degrees. — Observe moreover that the parity property depends on the choice of normal coordinate (cf. the general transformation rule [G2, (2.4.62)ff.]).

For an example where the result may be of interest, let us mention that the singular Green part of the solution operator for the Dirichlet or Neumann problem for a strongly elliptic differential operator of the form (3.62) has even-even parity for the coefficients c_{jk} in the Laguerre expansion of the symbol. (This is seen by calculations related to those in Remark 4.2 below, for fixed λ.) Then it has a canonical trace if n is even.

If we merely have that $\mathrm{tr}_n\, g$ has a parity that fits the dimension $n-1$ (instead of all c_{jk} having it), then there holds at least that $\mathrm{TR}(\mathrm{tr}_n\, G) = C_0(\mathrm{tr}_n\, G, S)$ is well-defined independently of S when n is even, resp. odd, by the result for closed manifolds ([KV], [G4]) applied to X'. This number can then be viewed as the nonlocal part of $C_0(G, P_{1,\mathrm{D}})$, in an explicit way.

Remark 3.16. The application of [GS1, pf. of Th. 2.1] to $\mathrm{tr}_n\, GR^N_\lambda$ shows even more, namely that in the parity cases (i) and (ii) in Theorem 3.15, all the coefficients $\tilde{c}_j$ with $\nu + n - 1 - j$ even and all the log-coefficients $\tilde{c}'_k$ with k even vanish in (1.10) (cf. also [G4, pf. of Th. 1.3]). This holds also for the pointwise kernel expansions as in (3.18).

Finally, let us mention another useful application of the method leading to Theorem 3.13:

Corollary 3.17. *When K is a Poisson operator of order ν and T is a trace operator of class 0 and order ν', then, for auxiliary operators $P_{1,\mathrm{D}}$ and S as above,*

$$C_0(KT,P_{1,\mathrm{D}})=C_0(TK,S)+\text{ \textit{local terms}}; \tag{3.65}$$

the local terms vanishing if $\nu+\nu'<1-n$ or $\notin\mathbb{Z}$.

Proof. Applying a partition of unity, we can reduce to the case where the operators are given on $\mathbb{R}^n_+$ (with compact x'-support). Here we write (cf. Lemma B.7)

$$K=\sum_{j\in\mathbb{N}}\Phi_jC_j,\quad T=\sum_{k\in\mathbb{N}}C'_k\Phi^*_k,$$

with rapidly decreasing sequences $(C_j)_{j\in\mathbb{N}}$ and $(C'_k)_{k\in\mathbb{N}}$ of ψdo's on $\mathbb{R}^{n-1}$ of order $\nu-\frac12$ resp. $\nu'+\frac12$ (and with compact x'-support). By (3.54) and (B.13),

$$\begin{aligned}C_0(KT,P_{1,\mathrm{D}})&=C_0(\sum_{j,k\in\mathbb{N}}\Phi_jC_jC'_k\Phi^*_k,P_{1,\mathrm{D}})\\&=C_0(\sum_jC_jC'_j,S)+\text{ local terms};\\C_0(TK,S)&=C_0(\sum_{j,k\in\mathbb{N}}C'_k\Phi^*_k\Phi_jC_j,S)=C_0(\sum_jC'_jC_j,S).\end{aligned}$$

The latter equals $C_0(\sum_jC_jC'_j,S)$ modulo local terms, by the commutativity property for ψdo's on $\mathbb{R}^{n-1}$ and the rapid decrease of the C_j and C'_j. This shows the identity (3.65), and the vanishing statement follows in the usual way. ☐

4. The integer order case with nonvanishing ψdo part

Consider now the case where A equals P_++G, of order $\nu\in\mathbb{Z}$. Here $C_0(P_++G,P_{1,\mathrm{D}})=C_0(P_+,P_{1,\mathrm{D}})+C_0(G,P_{1,\mathrm{D}})$, where $C_0(G,P_{1,\mathrm{D}})$ has already been analyzed. For the determination of $C_0(P_+,P_{1,\mathrm{D}})$, we rewrite as follows:

$$\begin{aligned}\operatorname{Tr}(P_+R^N_\lambda)&=\operatorname{Tr}(P_+Q^N_{\lambda,+})+\operatorname{Tr}(P_+G^{(N)}_\lambda)\\&=\operatorname{Tr}((PQ^N_\lambda)_+)-\operatorname{Tr}(G^+(P)G^-(Q^N_\lambda))+\operatorname{Tr}(P_+G^{(N)}_\lambda).\end{aligned} \tag{4.1}$$

It follows from the results of [GSc] recalled in Appendix B that the last two terms have expansions as in Proposition B.3, contributing only locally to $C_0(P_+,P_{1,\mathrm{D}})$. For the first term, we find the desired information by considering PQ^N_λ on $\widetilde{X}$. As in Section 3 (see the details after Lemma 3.1), we can use local coordinates and a subordinate parition of unity, now covering all of $\widetilde{X}$ with open sets U_i, such that subsets of X resp. $\widetilde{X}\setminus X$ are mapped to subsets of $\overline{\mathbb{R}}^n_+$ resp. $\mathbb{R}^n_-$, the intersections of the U_i with X' being mapped into $\partial\overline{\mathbb{R}}^n_+$. Again we can replace the images $V_{j(i_1,i_2)}$ by sets V'_{i_1,i_2} with positive distance from one another (in the x'-direction), so that we can refer to one localized operator. By [GS1,

Th. 2.1 and 2.7] and the more detailed information in [G4, Th. 1.3 ff.], the kernel of PQ_λ^N in the localized situation has a diagonal expansion:

$$K(PQ_\lambda^N,x,x) \sim \sum_{j\in\mathbb{N}} \tilde c_j(x)(-\lambda)^{\frac{\nu+n-j}{2}-N} + \sum_{k\in\mathbb{N}} \big(\tilde c_k'(x)\log(-\lambda)+\tilde c_k''(x)\big)(-\lambda)^{-k-N}, \tag{4.2}$$

where

$$c_{\nu+n}(x)+c_0''(x) = ⨍ p(x,\xi)\, đ\xi + \text{ local terms.} \tag{4.3}$$

From the sets mapped into $\overline{\mathbb{R}}^n_+$, we find $\operatorname{Tr}_X((PQ_\lambda^N)_+)$ by integrating over $\mathbb{R}^n_+$. Then (4.2) implies

$$\operatorname{Tr}_X((PQ_\lambda^N)_+) \sim \sum_{j\in\mathbb{N}} \tilde c_{j,+}(-\lambda)^{\frac{\nu+n-j}{2}-N} + \sum_{k\in\mathbb{N}} \big(\tilde c_{k,+}'\log(-\lambda)+\tilde c_{k,+}''\big)(-\lambda)^{-k-N}, \tag{4.4}$$

where the coefficients are obtained from the coefficients in (4.2) by integration in x. In particular, in view of (4.3),

$$c_{\nu+n,+}+c_{0,+}'' = \int_{\mathbb{R}^n_+} ⨍ \operatorname{tr} p(x,\xi)\, đ\xi dx + \text{ local terms.} \tag{4.5}$$

Here $c_{\nu+n,+}$ is also locally determined, so we can conclude:

Theorem 4.1. *Let P be of order $\nu\in\mathbb{Z}$, having the transmission property at X'. In the localized situation,*

$$C_0(P_+,P_{1,\mathrm{D}}) = \int_{\mathbb{R}^n_+} ⨍ \operatorname{tr} p(x,\xi)\, đ\xi dx + \textit{ local terms.} \tag{4.6}$$

When P_1 and P_2 are two choices of auxiliary operator, then $C_0(P_+,P_{1,\mathrm{D}})-C_0(P_+,P_{2,\mathrm{D}})$ is locally determined.

Proof. It only remains to establish the last statement: It holds, since the global term $\int_{\mathbb{R}^n_+} ⨍ \operatorname{tr} p(x,\xi)\, đ\xi dx$ cancels out in the calculation of the difference in local coordinates, leaving only locally determined terms. □

Remark 4.2. As usual, we can ask for cases where the local terms vanish. They do so for $\nu<-n$, where

$$C_0(P_+,P_{1,\mathrm{D}}) = \operatorname{Tr} P_+$$

is easily seen. For $\nu\ge -n$, one can analyze cases with parity properties, but they will in general not have vanishing local contributions, as the following considerations show: Let $P_+=I$, the simplest possible choice, and let P_1 be of the simple form (3.62) with P_1' selfadjoint positive. We have to expand

$$\operatorname{Tr}(IR_\lambda^N) = \operatorname{Tr}(Q_{\lambda,+}^N) + \operatorname{Tr}(G_\lambda^{(N)}).$$

Since Q_λ is even-even,

$$\operatorname{Tr}(Q^N_{\lambda,+}) \sim \sum_{j'\in\mathbb{N}} \tilde{c}_{2j',+}(-\lambda)^{\frac{n}{2}-j'-N},$$

which contributes to the power $(-\lambda)^{-N}$ when n is *even*. On the cylinder $X'\times\overline{\mathbb{R}}_+$, the Dirichlet s.g.o. G_λ can be explicitly constructed (cf. (2.10)) to be

$$G_\lambda = -K_{A_\lambda}(2A_\lambda)^{-1}T_{A_\lambda}, \text{ where}$$
$$A_\lambda = (P'_1-\lambda)^{\frac12},\ K_{A_\lambda}: v(x')\mapsto e^{-x_nA_\lambda}v,\ T_{A_\lambda}: u(x)\mapsto \int_0^\infty e^{-x_nA_\lambda}u(x',x_n)\,dx_n;$$

with notation as in [G6]. Then $\operatorname{tr}_n G_\lambda = -(2A_\lambda)^{-2} = -\frac14(P'_1-\lambda)^{-1}$. This is a resolvent on X', and the N'th derivative has a trace expansion

$$\operatorname{Tr}(G^{(N)}_\lambda) = \operatorname{Tr}_{X'}(\operatorname{tr}_n G^{(N)}_\lambda) \sim \sum_{j'\in\mathbb{N}} a_{2j'}(-\lambda)^{\frac{n-1}{2}-j'-N},$$

which contributes to the power $(-\lambda)^{-N}$ when n is *odd.*

However, in cases where p has a parity that fits with the dimension n, one can at least observe that the integral $\int_{\mathbb{R}^n_+} ⨍ \operatorname{tr} p(x,\xi)\, đ\xi dx$ has a coordinate invariant meaning (cf. Remark 3.2), so it gives an explicit value that can be considered as the nonlocal part of $C_0(P_+,P_{1,\mathrm{D}})$.

Theorem 4.1 assures the validity of (2.7) for $C_0(P_++G,P_{1,\mathrm{D}})$. In the proof of (2.8), we restrict the attention to operators P of normal order ≤ 0.

For (2.8), it would be possible to refer again to Laguerre expansions (as in the considerations of s.g.o.s in Section 3), using that P on the symbol level acts like a Toeplitz operator. More precisely, if one writes $p(x',0,\xi) = \sum_{j\in\mathbb{Z}} a_j(x',\xi')\hat\psi_j(\xi_n,\sigma)$ with

$$\hat\psi_j(\xi_n,\sigma) = \frac{(\sigma-i\xi_n)^j}{(\sigma+i\xi_n)^j}, \tag{4.7}$$

then in the one-dimensional calculus based on Laguerre expansion in $L_2(\mathbb{R}_+)$, $p(x',0,\xi',D_n)_+$ acts like the Toeplitz operator with matrix $(a_{j-k})_{j,k\in\mathbb{N}}$, cf. [G2, Rem. 2.2.12]. With some effort, the localness of $C_0([P_+,G'],P_{1,\mathrm{D}})$ and $C_0([P_+,P'_+],P_{1,\mathrm{D}})$ can be proved by use of the localness of C_0 on commutators over X', but the interpretations and composition rules for the involved operators are not altogether simple to deal with. Instead we shall rely on repeated commutator techniques.

In order to study the commutator of a pseudodifferential operator P_+ of order $\nu\in\mathbb{Z}$ with an operator $A' = P'_+ + G'$ of order $\nu'\in\mathbb{Z}$, we write:

$$\operatorname{Tr}([P_+,A']R^N_\lambda) = \operatorname{Tr}([P_+,A']Q^N_{\lambda,+}) + \operatorname{Tr}([P_+,A']G^{(N)}_\lambda). \tag{4.8}$$

We know from Proposition B.3 that the second term contributes only locally to C_0, and go on with an analysis the first term:

$$\begin{aligned}\operatorname{Tr}([P_+,A']Q^N_{\lambda,+}) &= \operatorname{Tr}(A'Q^N_{\lambda,+}P_+) - \operatorname{Tr}(A'P_+Q^N_{\lambda,+})\\ &= \operatorname{Tr}(A'[Q^N_\lambda,P]_+) - \operatorname{Tr}(A'G^+(Q^N_\lambda)G^-(P)) + \operatorname{Tr}(A'G^+(P)G^-(Q^N_\lambda))\\ &= \operatorname{Tr}(A'[Q^N_\lambda,P]_+) - \operatorname{Tr}(G^-(P)A'G^+(Q^N_\lambda)) + \operatorname{Tr}(A'G^+(P)G^-(Q^N_\lambda)).\end{aligned} \tag{4.9}$$

Again, the last two terms contribute only locally to C_0, by Proposition B.3. It is used that our hypotheses on P assure that the s.g.o.s $G^{\pm}(P)$ are of class 0.

Now consider the remaining term $\operatorname{Tr}(A'[Q_\lambda^N, P]_+)$. The expression $[Q_\lambda^N, P]_+$ is not very convenient in the boundary calculus, as a mixture of strongly polyhomogeneous parameter-dependent and arbitrary parameter-independent interior operators. However, we shall now show that it can be reduced to an expression with λ-dependent factors to the right only, with better decrease in λ than Q_λ, plus a manageable remainder term.

Lemma 4.3. *Let P be of order ν and normal order ≤ 0. For $r > 0$, let $P_{(r)}$ denote the r'th commutator of P with P_1:*

$$P_{(1)} = [P, P_1],\ P_{(2)} = [[P, P_1], P_1],\ \dots,\ P_{(r)} = [\cdots[[P, P_1], P_1]\dots, P_1]; \tag{4.10}$$

it is of order $\nu + r$ and of normal order $\le \min\{\nu + r, 2r\}$).

For any $M > 0$, $[Q_\lambda, P]$ may be written as

$$[Q_\lambda, P] = P_{(1)}Q_\lambda^2 + P_{(2)}Q_\lambda^3 + \dots + P_{(M)}Q_\lambda^{M+1} + Q_\lambda P_{(M+1)}Q_\lambda^{M+1}. \tag{4.11}$$

Proof. Since P_1 has scalar principal symbol and is of order 2, $[P, P_1]$ is of order $\nu + 1$. Since P is of normal order 0 and P_1 is of normal order 2, $[P, P_1]$ is of normal order 2 (e.g., $\partial_{x_1} p_{1,2}$ is generally so). Similarly, every subsequent commutation lifts the order by 1 and the normal order by 2, so the r'th commutator is of order $\nu + r$ and normal order $\le 2r$. Since it has the transmission property, it also has normal order $\le \nu + r$. This explains the first statement. The second statement follows by successive applications of the following calculation:

$$\begin{aligned}[][Q_\lambda, P_{(r)}] &= Q_\lambda P_{(r)} - P_{(r)}Q_\lambda = Q_\lambda P_{(r)}(P_1 - \lambda)Q_\lambda - Q_\lambda(P_1 - \lambda)P_{(r)}Q_\lambda \\ &= Q_\lambda[P_{(r)}, P_1]Q_\lambda = Q_\lambda P_{(r+1)}Q_\lambda = P_{(r+1)}Q_\lambda^2 + [Q_\lambda, P_{(r+1)}]Q_\lambda;\end{aligned}$$

hence

$$\begin{aligned}[][Q_\lambda, P] &= P_{(1)}Q_\lambda^2 + [Q_\lambda, P_{(1)}]Q_\lambda = \dots \\ &= P_{(1)}Q_\lambda^2 + P_{(2)}Q_\lambda^3 + \dots + P_{(M)}Q_\lambda^{M+1} + [Q_\lambda, P_{(M)}]Q_\lambda^M \\ &= P_{(1)}Q_\lambda^2 + P_{(2)}Q_\lambda^3 + \dots + P_{(M)}Q_\lambda^{M+1} + Q_\lambda P_{(M+1)}Q_\lambda^{M+1}. \quad \square\end{aligned}$$

Proposition 4.4. *Let P and P' be ψdo's of order ν resp. $\nu' \in \mathbb{Z}$ and normal order ≤ 0, and let G' be an s.g.o. of order ν' and class 0; denote $P'_+ + G' = A'$. For $N > (\nu + \nu' + n)/2$, $\operatorname{Tr}(A'[Q_\lambda^N, P]_+)$ has an expansion*

$$\operatorname{Tr}(A'[Q_\lambda^N, P]_+) \sim \sum_{j\ge 0} a_j(-\lambda)^{\frac{n+\nu+\nu'-j}{2} - N} + \sum_{k\ge 1}(a'_k \log(-\lambda) + a''_k)(-\lambda)^{-\frac{k}{2} - N}. \tag{4.12}$$

Proof. We know from the preceding calculation that there is an expansion as in (1.10) with ν replaced by $\nu + \nu'$; the point is to show that the series in k starts with a lower power than $-N$.

Recalling that $Q_\lambda^N = \frac{1}{(N-1)!}\partial_\lambda^{N-1} Q_\lambda$, we find from (4.11):

$$[Q_\lambda^N, P] = c_1 P_{(1)} Q_\lambda^{N+1} + c_2 P_{(2)} Q_\lambda^{N+2} + \cdots + c_M P_{(M)} Q_\lambda^{N+M} + \sum_{1\le l\le N} c_l' Q_\lambda^l P_{(M+1)} Q_\lambda^{N+M+1-l}. \tag{4.13}$$

In order to establish the expansion (4.12) we shall study the terms

$$P'_+(P_{(r)}Q_\lambda^{(N+r)})_+ \text{ and } G'(P_{(r)}Q_\lambda^{(N+r)})_+;\quad 1\le r\le M; \tag{4.14}$$

as well as

$$A'(Q_\lambda^l P_{(M+1)} Q_\lambda^{N+M+1-l})_+;\quad 1\le l\le N. \tag{4.15}$$

The operator $Q_\lambda^l P_{(M+1)} Q_\lambda^{N+M+1-l}$ is a ψdo on $\widetilde{X}$ with a symbol in $S^{\nu+M+1,0,-2N-2M-2}$. The compositions in (4.15) will therefore be trace-class on $L^2(X)$ when $\nu - M - 2N - 1 \le 0$, $\nu' + \nu - M - 2N - 1 < -n$, and their trace will for any given M' be $O(\lambda^{-M'})$ for sufficiently large M. Our assertion will thus be true for these terms.

We turn to the terms in (4.14), considered in local coordinates. In view of (2.6), we can decompose $P_{(r)} = P'_{(r)} + P''_{(r)}$, where $P''_{(r)}$ is a differential operator of order $\le \nu + r$ and normal order $\le \min\{\nu + r, 2r\}$, and $P'_{(r)}$ has the property that $G^+(P'_{(r)})$ is of class 0, as a sum of an operator of normal order 0 and an operator with a factor $x_n^{M'}$ to the right, $M' \ge 2r$. Then

$$(P_{(r)}Q_\lambda^{N+r})_+ = P'_{(r),+}Q_{\lambda,+}^{N+r} + G^+(P'_{(r)})G^-(Q_\lambda^{N+r}) + (P''_{(r)}Q_\lambda^{N+r})_+. \tag{4.16}$$

Note that

$$P''_{(r)} = \sum_{0\le j\le 2r} S_j(x, D_{x'}) D_{x_n}^j$$

with tangential differential operators S_j of order $\nu + r - j$. The symbol of Q_λ^{N+r} has the structure described in (A.10), with typical term $r_{N+r,J,m}(p_{1,2}+\mu^2)^{-m}$ where $J+2N+2r \le 2m \le 4J + 2N + 2r$ and $r_{N+r,J,m}$ is polynomial in ξ of degree $\le 2m - 2N - 2r - J$. When this is multiplied by ξ_n^{2r} and we redefine $J+2r = J'$, we get a term with numerator of order $\le 2m - 2N - J'$ and denominator $(p_{1,2}+\mu^2)^m$, $J' + 2N \le 2m \le 4J' + 2N - 6r \le 4J' + 2N$; this is of the type in the symbol expansion of Q_λ^N. Similarly, multiplication by ξ_n^j for $j \le 2r$ gives terms of the types in the symbol expansions of $Q_\lambda^{N'}$ with $N' \ge N$. The point of this analysis is that we can write

$$G^-(P''_{(r)}Q_\lambda^{N+r}) = \sum_{0\le j\le 2r} S_j(x, D_{x'}) G^-(D_{x_n}^j Q_\lambda^{N+r}), \tag{4.17}$$

where the factors $G^-(D_{x_n}^j Q_\lambda^{N+r})$ have structures like the $G^-(Q_\lambda^{N'})$, $N' \ge N$, so that Proposition B.3 can be applied.

The composition of A' with the terms in (4.16) gives a number of terms:

$$\begin{aligned}
&\text{(i) } P'_+P'_{(r),+}Q^{N+r}_{\lambda,+} = (P'P'_{(r)})_+Q^{N+r}_{\lambda,+} - G^+(P')G^-(P'_{(r)})Q^{N+r}_{\lambda,+}\\
&\quad = (P'P'_{(r)}Q^{N+r}_\lambda)_+ - G^+(P'P'_{(r)})G^-(Q^{N+r}_{\lambda,+}) - G^+(P')G^-(P'_{(r)})Q^{N+r}_{\lambda,+},\\
&\text{(ii) } G'P'_{(r),+}Q^{N+r}_{\lambda,+},\\
&\text{(iii) } A'G^+(P'_{(r)})G^-(Q^{N+r}_\lambda),\\
&\text{(iv) } P'_+(P''_{(r)}Q^{N+r}_\lambda)_+ = (P'P''_{(r)}Q^{N+r}_\lambda)_+ - G^+(P')G^-(P''_{(r)}Q^{N+r}_\lambda),\\
&\text{(v) } G'(P''_{(r)}Q^{N+r}_\lambda)_+.
\end{aligned}$$

To the terms in (i) and (iii) with the s.g.o.-factor $G^-(Q^{r+N}_\lambda)$ we can apply Proposition B.3 with ν, N replaced by $\nu+\nu', N+r$; here the series in k starts with the power $(-\lambda)^{-\frac12-N-r}$.

The term $G^+(P')G^-(P''_{(r)}Q^{N+r}_\lambda)$ in (iv) with the s.g.o.-factor $G^-(P''_{(r)}Q^{r+N}_\lambda)$ is written in view of (4.17) as

$$G^+(P')G^-(P''_{(r)}Q^{N+r}_\lambda) = \sum_{0\le j\le 2r} G^+(P')S_j(x,D_{x'})G^-(D^j_{x_n}Q^{N+r}_\lambda),$$

where each $G^+(P')S_j(x,D_{x'})$ is of class 0. Proposition B.3 applies and gives contributions as in (4.12).

For the terms in (i) and (ii) of the form of an s.g.o. composed with $Q^{r+N}_{\lambda,+}$, we apply Proposition B.5. Since $r\ge 1$, the series in k has only powers $\le -1-N$ of λ, so the contribution to the coefficient of $(-\lambda)^{-N}$ is local.

For (v), we rewrite $G'(P''_{(r)}Q^{N+r}_\lambda)_+ = (G'P''_{(r),+})Q^{N+r}_{\lambda,+}$, noting that the leftover term for the product is zero, since $P''_{(r)}$ is a differential operator. Now $G'P''_{(r),+}$ is an s.g.o. of order $\nu'+\nu+r$ and class $\le \min\{\nu+r, 2r\}$. It therefore has a representation

$$G_0 + \sum_{0\le j\le 2r-1} K_j\gamma_j$$

with an s.g.o. G_0 of order $\nu+\nu'+r$ and class 0 and Poisson operators K_j of order $\nu+\nu'+r-j$. The composition $G_0Q^{N+r}_{\lambda,+}$ is like (ii); Proposition B.5 applies to give an expansion (4.12).

Next we decompose K_j according to Lemma B.7: $K_j = \sum_{l\ge 0}\Phi_l C_{jl}$ with a rapidly decreasing sequence $(C_{jl})_l$ in $S^{\nu+\nu'+r-j+\frac12}$. Then

$$\operatorname{Tr}(K_j\gamma_jQ^{N+r}_{\lambda,+}) = \sum_{l\ge 0}\operatorname{Tr}_{\mathbb{R}^{n-1}}(\gamma_0\partial^j_{x_n}Q^{N+r}_{\lambda,+}\Phi_lC_{jl}).$$

The composition $\gamma_0\partial^j_{x_n}Q^{N+r}_{\lambda,+}\Phi_l$ is a ψdo on $\mathbb{R}^{n-1}$ with the symbol

$$\sum_{j_1+j_2=j}\binom{j}{j_1}\int^+ (i\xi_n)^{j_1}\,\partial^{j_2}_{x_n}q^{\circ(N+r)}(x',0,\xi,\mu)\,\hat\varphi_l(\xi_n,\sigma)đ\xi_n. \tag{4.18}$$

We know from (A.8)–(A.11) that $(i\xi_n)^{j_1}\partial_{x_n}^{j_2} q^{\circ(N+r)}_{-2(N+r)-J}$ is a finite sum of terms of the form $\xi_n^{j_1} r_{J,m}(p_{12}+\mu^2)^{-m}$ with a polynomial $r_{J,m}$ of degree $\le 2m-2(N+r)-J$. Decomposing it into simple fractions, we obtain a sum of terms of the form $r^{\pm}_{J,k}(\kappa^{\pm}\pm i\xi_n)^{-k}$ with $r^{\pm}_{J,k}(x',\xi',\mu)$ strongly homogeneous in (ξ',μ) of degree $\le j+k-2(N+r)-J$. Inserting this into (4.18) and using Lemma B.1, we see that $\gamma_j Q^{N+r}_{\lambda,+}\Phi_l$ has its symbol in $S^{\frac12,0,-2N-2r+j}$; the symbol seminorms grow at most polynomially in l. Since $j<2r$, we then reach the expansion (4.12).

The remaining ψdo terms in (i) and (iv), $(P'P'_{(r)}Q^{N+r}_\lambda)_+$ and $(P'P''_{(r)}Q^{N+r}_\lambda)_+$, add up to $(P'P_{(r)}Q^{N+r}_\lambda)_+$, which is treated as in Theorem 4.1, using the pointwise kernel expansion on $\widetilde{X}$. Here $P'P_{(r)}Q^{r+N}_\lambda$ has symbol in $S^{\nu'+\nu+r,0,-2r-2N}\subset S^{\nu'+\nu-r-2N,0,0}$ $\cap S^{\nu'+\nu+r,-2r-2N,0}$ with d-index $\le -2-2N$ since $r\ge 1$, so we get an expansion contributing only locally to the coefficient of $(-\lambda)^{-N}$. $\square$

We now have all the ingredients for the proof of:

Theorem 4.5. *Let P and P' be ψdo's of order ν resp. $\nu'\in\mathbb{Z}$ and normal order 0, and let G and G' be singular Green operators of order ν resp. ν' and class 0. Then $C_0([P_++G,P'_++G'],P_{1,\mathrm{D}})$ is locally determined.*

Proof. We have that

$$[P_++G,P'_++G']=[P_++G,P'_+]+[P_+,G']+[G,G'].$$

The last term was shown in Theorem 3.13 to contribute locally to C_0. For the two other terms we have the analysis above, through (4.8), (4.9) and Proposition 4.4, showing that they contribute only locally to C_0. $\square$

This completes the proof that (2.8) holds also for operators of type $A=P_++G$, so $C_0(P_++G,P_{1,\mathrm{D}})$ is indeed a quasi-trace on such operators. In addition, we have found the interesting information that the contribution from P_+ can be traced back to a pointwise defined contribution from P over $\widetilde{X}$, and that the contribution from G can be traced back to an interior trace contribution of order $-\infty$ plus a contribution from the normal trace on X'; here both $\widetilde{X}$ and X' are compact manifolds without boundary.

Appendix A. The structure of the auxiliary operators

We here recall the symbol formulas established and used in [GSc], and some useful consequences. The parameter-dependent entries were indexed by $\mu=(-\lambda)^{\frac12}$ in [GSc]; we simply replace this here by indexation by λ, although μ can still appear as a variable.

Throughout this paper, we denote by $[\xi']$ a positive C^∞ function of ξ' that coincides with $|\xi'|$ for $|\xi'|\ge 1$. It will often be denoted $\sigma(\xi')$ or just σ.

The principal (second-order) symbol of P_1 is denoted $p_{1,2}$, so the principal symbol of Q_λ is $q_{-2}=(p_{1,2}-\lambda)^{-1}=(p_{1,2}+\mu^2)^{-1}$.

We assume P_1 to be strongly elliptic; this means that the principal symbol $p_{1,2}(x,\xi)$ has positive real part when $\xi\ne 0$. Since $|\operatorname{Im} p_{1,2}(x,\xi)|\le |p_{1,2}(x,\xi)|\le C\operatorname{Re} p_{1,2}(x,\xi)$, there is a sector Γ such that $p_{1,2}(x,\xi)+\mu^2\ne 0$ when $\mu\in\Gamma\cup\{0\}$, $(\xi,\mu)\ne(0,0)$; cf. also (3.9).

The following observation will be useful:

Lemma A.1. *Let $p(x,\xi)$ be a uniformly strongly elliptic homogeneous second-order differential operator symbol on $\mathbb{R}^n$, with $p-\lambda = p+\mu^2$ invertible for $\mu \in \Gamma$. Then for any m, $L>0$,*

$$
\begin{aligned}
(p+\mu^2)^{-m} &= \sum_{0\le j<L} c_{m,j}\mu^{-2m-2j}p^j + \mu^{-2m-2L}p'_L(x,\xi,\mu), \text{ where}\\
p'_L(x,\xi,\mu) &= \sum_{0\le k\le m} c_{m,L,k}p^{L+k}(p+\mu^2)^{-k}.
\end{aligned}
\tag{A.1}
$$

Here one has for all indices:

$$
|\partial_\lambda^N \partial_x^\beta \partial_\xi^\alpha p'_L(x,\xi,\mu)| \le C_{\alpha,\beta,N}\langle\xi\rangle^{2L-|\alpha|-2N},
\tag{A.2}
$$

with constants $C_{\alpha,\beta,N}$ independent of $\mu\in\Gamma$ for $|\mu|\ge 1$.

Proof. One has for $b \in \mathbb{C}\setminus\{1\}$, $L\ge 1$,

$$
\begin{aligned}
(1-b)^{-m} &= \frac{1}{(m-1)!}\partial_b^{m-1}\frac{1}{1-b} = \frac{1}{(m-1)!}\partial_b^{m-1}\Big(\sum_{0\le k<L+m-1} b^k + \frac{b^{L+m-1}}{1-b}\Big)\\
&= \sum_{0\le j<L} c_j b^j + \sum_{1\le l\le m}\frac{c'_l b^{L+l-1}}{(1-b)^l}.
\end{aligned}
\tag{A.3}
$$

An application to $1-b = 1+p/\mu^2$ gives

$$
\begin{aligned}
(p+\mu^2)^{-m} &= \mu^{-2m}(1+p/\mu^2)^{-m}\\
&= \mu^{-2m}\big(\sum_{0\le j<L} c_j\mu^{-2j}p^j + \sum_{1\le l\le m} c'_l\mu^{-2L-2l+2}p^{L+l-1}(1+p/\mu^2)^{-l}\big)\\
&= \sum_{0\le j<L} c_j\mu^{-2m-2j}p^j + \mu^{-2m-2L+2}\sum_{1\le l\le m} c'_l p^{L+l-1}(p+\mu^2)^{-l}.
\end{aligned}
$$

Insertion of $\mu^2(p+\mu^2)^{-1} = 1-p(p+\mu^2)^{-1}$ in the last sum leads to (A.1). Since p is polynomial of degree 2, $(p+\mu^2)^{-1}$ is C^∞ and homogeneous in (ξ,μ) of degree -2 for $(\xi,\mu)\neq 0$, and $\partial_\lambda^N(p-\lambda)^{-l} = c_{l,N}(p-\lambda)^{-l-N}$,

$$
\begin{aligned}
|\partial_\lambda^N \partial_x^\beta \partial_\xi^\alpha p'_L(x,\xi,\mu)| \le C'_{\alpha,\beta,N} &\sum_{0\le k\le m,\alpha'\le\alpha} (\partial_\xi^{\alpha'}p^{L+k})|(\xi,\mu)|^{-2k-2N-|\alpha-\alpha'|}\\
&\le C_{\alpha,\beta,N}\langle\xi\rangle^{2L-|\alpha|-2N}
\end{aligned}
$$

for $|\mu|\ge 1$, showing (A.2). □

Let $\xi = (\xi',\xi_n)$ in local coordinates at the boundary, then for $(\xi',\mu)\neq(0,0)$, the strong ellipticity implies that the polynomial in ξ_n

$$
p_{1,2}(x',0,\xi',\xi_n)+\mu^2 = a(x')\xi_n^2 + b(x',\xi')\xi_n + c(x',\xi') + \mu^2
\tag{A.4}
$$

has two roots $\varrho_1(x',\xi',\mu)$ and $\varrho_2(x',\xi',\mu)$ in $\mathbb{C}\setminus\mathbb{R}$. When $\mu\in\overline{\mathbb{R}}_+$, one of the roots, say ϱ_1, lies in $\mathbb{C}_+$ and the other, ϱ_2, in $\mathbb{C}_-$. For, $a\xi_n^2+b\xi_n+c+\mu^2$ can be carried into $\operatorname{Re}a\,\xi_n^2+\operatorname{Re}b\,\xi_n+\operatorname{Re}c+\mu^2$ by a homotopy that preserves the property of having positive real part, and for the latter polynomial, the roots are placed in this way; they depend continuously on the polynomial, hence cannot cross the real axis. The placement of the roots is also preserved when μ is moved to a general element of Γ. Thus we can denote the roots $\pm i\kappa^\pm(x',\xi',\mu)$, where $\kappa^\pm$ have positive real part; they depend smoothly on (x',ξ',μ) for $(\xi',\mu)\neq(0,0)$. They are homogeneous of degree 1 in (ξ',μ) and bounded away from $\mathbb{R}$ for $|(\xi',\mu)|=1$, $\mu\in\Gamma\cup\{0\}$, so in fact they take values in a sector $\{|\arg z|\le\frac{\pi}{2}-\delta\}$ for some $\delta>0$.

We recall from [GSc, Sect. 2.b] (with some small precisions):

Lemma A.2. *The symbol of Q_λ has the following form in local coordinates, for $\mu\in\Gamma$:*

$$
\begin{aligned}
q(x,\xi,\mu)&\sim\sum_{l\in\mathbb{N}}q_{-2-l}(x,\xi,\mu),\ \text{with}\\
q_{-2}(x,\xi,\mu)&=(p_{1,2}(x,\xi)+\mu^2)^{-1}I,\\
q_{-2-J}(x,\xi,\mu)&=\sum_{J/2+1\le m\le 2J+1}\frac{r_{J,m}(x,\xi)}{(p_{1,2}(x,\xi)+\mu^2)^m},\ \text{for } J\ge 0;
\end{aligned}
\tag{A.5}
$$

here the $r_{J,m}$ are $(\dim E\times\dim E)$-matrices of homogeneous polynomials in ξ of degree $2m-2-J$ with smooth coefficients, and the remainders $q'_{-2-M}=q-\sum_{0\le J<M}q_{-2-J}$ satisfy estimates for all indices $\partial_x^\beta\partial_\xi^\alpha\partial_\mu^k q'_{-2-M}=O(\langle\xi',\mu\rangle^{-2-M-|\alpha|-k})$, for $|\mu|\ge1$, μ in closed subsectors of Γ.

Concerning $q(x',0,\xi)$, we have: Writing

$$
\begin{aligned}
p_{1,2}(x',0,\xi)+\mu^2&=a(x')(\xi_n-i\kappa^+(x',\xi',\mu))(\xi_n+i\kappa^-(x',\xi',\mu))\\
&=a(x')(\kappa^+(x',\xi',\mu)+i\xi_n)(\kappa^-(x',\xi',\mu)-i\xi_n),
\end{aligned}
\tag{A.6}
$$

we can decompose each term in simple fractions (at $x_n=0$):

$$
\begin{aligned}
\frac{r_{J,m}(x',0,\xi)}{(p_{1,2}(x',\xi)+\mu^2)^m}&=h^+\frac{r_{J,m}}{(p_{1,2}+\mu^2)^m}+h^-\frac{r_{J,m}}{(p_{1,2}+\mu^2)^m},\\
h^\pm\frac{r_{J,m}}{(p_{1,2}+\mu^2)^m}&=\sum_{1\le j\le m}\frac{r^\pm_{J,m,j}(x',\xi',\mu)}{(\kappa^\pm(x',\xi',\mu)\pm i\xi_n)^j},
\end{aligned}
\tag{A.7}
$$

where the $r^\pm_{J,m,j}(x',\xi',\mu)$ are strongly homogeneous of degree $j-J-2$ in (ξ',μ). This gives a decomposition of $q(x',0,\xi',\mu)$:

$$
q=h^+q+h^-q=q^++q^-,\quad q^\pm(x',\xi,\mu)\sim\sum_{J\ge0}q^\pm_{-2-J}(x',\xi,\mu),
$$

with terms as in (A.7).

The structure of the normal derivatives is similar: We have

$$
\partial_{x_n}^l q_{-2-J}(x,\xi,\mu)=\sum_{1+J/2\le m\le 2J+1+l}\frac{r^l_{J,m}(x,\xi)}{(p_{1,2}(x,\xi)+\mu^2)^m},
$$

for all l, with homogeneous polynomials $r^l_{J,m}(x,\xi)$ of degree $2m-2-J$ in ξ. At $x_n=0$ we can decompose as before, obtaining

$$
\begin{aligned}
&\partial^l_{x_n} q = \partial^l_{x_n} q^+ + \partial^l_{x_n} q^-, \quad \partial^l_{x_n} q^\pm(x',\xi,\mu) \sim \sum_{J\ge 0} \partial^l_{x_n} q^\pm_{-2-J}(x',\xi,\mu),\\
&\partial^l_{x_n} q_{-2-J}(x',\xi,\mu) = \partial^l_{x_n} q^+_{-2-J}(x',\xi,\mu) + \partial^l_{x_n} q^-_{-2-J}(x',\xi,\mu),\\
&\partial^l_{x_n} q^\pm_{-2-J}(x',\xi,\mu) = \sum_{1\le j\le 2J+1+l} \frac{r^{l,\pm}_{J,j}(x',\xi',\mu)}{(\kappa^\pm(x',\xi',\mu)\pm i\xi_n)^j},
\end{aligned}
\tag{A.8}
$$

where the numerators $r^{l,\pm}_{J,j}(x',\xi',\mu)$ are strongly homogeneous of degree $j-J-2$.

It is useful to observe that the $r^\pm_{J,m,j}$ as well as $\kappa^\pm$ are in fact functions of (x',ξ',λ), $\lambda=-\mu^2$, strongly quasihomogeneous in (ξ',λ) with weight $(1,2)$ in the following sense: We say that $r(x',\xi',\lambda)$ is $(1,2)$-homogeneous in (ξ',λ) of degree d, when

$$
r(x',t\xi',t^2\lambda) = t^d r(x',\xi',\lambda);
\tag{A.9}
$$

it is strongly so if (A.9) holds for $|\xi'|+|\lambda|\ge\varepsilon$, weakly so if it holds for $|\xi'|\ge\varepsilon$. Then since $r(x',\xi',\lambda)=t^{-d}r(x',t\xi',t^2\lambda)$, it is readily checked that $\partial^N_\lambda r$ is $(1,2)$-homogeneous of degree $d-2N$. Thus for the symbols that depend on μ through λ in this way, differentiation with respect to λ lowers the homogeneity degree in (ξ',μ) by two steps, preserving strong homogeneity. The estimates of λ-derivatives of the remainders q'_{-2-M} likewise improve by two steps for each derivative.

These considerations of λ-derivatives, playing on the strong quasi-homogeneity of symbols coming from R_λ, will replace the calculations for higher powers of R_λ used in [GSc], cf. (2.12). (The calculus in [GH] is set up to handle the anisotropic homogeneity in terms of ξ' and λ and could give further information on higher terms in the full trace expansions, but for the discussion of the leading nonlocal term we carry out here, the calculus of [GS1] will suffice.) For convenience, we recall explicitly the structure of the formulas for symbols connected with higher powers of Q_λ, denoting the symbol of $Q^N_\lambda = \frac{\partial^{N-1}_\lambda}{(N-1)!}Q_\lambda$ by $q^{\circ N}$:

$$
\begin{aligned}
&q^{\circ N}(x,\xi,\mu) \sim \sum_{l\in\mathbb{N}} q^{\circ N}_{-2N-l}(x,\xi,\mu), \text{ with } q^{\circ N}_{-2N} = (p_{1,2}+\mu^2)^{-N},\\
&q^{\circ N}_{-2N-J}(x,\xi,\mu) = \sum_{J/2+N\le m\le 2J+N} \frac{r_{N,J,m}(x,\xi)}{(p_{1,2}(x,\xi)+\mu^2)^m}, \text{ for } J\ge 0,
\end{aligned}
\tag{A.10}
$$

where the $r_{N,J,m}$ are homogeneous polynomials in ξ of degree $2m-2N-J$ with smooth coefficients. Moreover for $x=(x',0)$,

$$
\begin{aligned}
&q^{\circ N} = h^+ q^{\circ N} + h^- q^{\circ N} = q^{\circ N,+} + q^{\circ N,-},\\
&q^{\circ N,\pm}(x',\xi,\mu) \sim \sum_{J\ge 0} q^{\circ N,\pm}_{-2-J}(x',\xi,\mu),\\
&q^{\circ N,\pm}_{-2-J}(x',\xi,\mu) = \sum_{1\le j\le 2J+N} \frac{r^\pm_{N,J,j}(x',\xi',\mu)}{(\kappa^\pm(x',\xi',\mu)\pm i\xi_n)^j};
\end{aligned}
\tag{A.11}
$$

here $r^{\pm}_{N,J,j}(x',\xi',\mu)$ are strongly homogeneous of degree $j-J-2N$ in (ξ',μ).

For $\partial^l_{x_n} q^{\circ N}(x,\xi,\mu)$ and $\partial^l_{x_n} q^{\circ N,\pm}(x',\xi,\mu)$, we obtain corresponding results: Their structure is as in (A.10) and (A.11), except that now the summation will be over the sets $J/2+N\le m\le 2J+N+l$ and $1\le j\le 2J+N+l$.

The following symbols derived from the principal symbol of q at $x_n=0$ played an important role in [GSc, Sect. 5]:

$$
\begin{aligned}
\alpha^{(1)}(x',\xi',\mu) &= [h^+q_{-2}(x',0,\xi,\mu)]_{\xi_n=-i\sigma} + [h^-q_{-2}(x',0,\xi,\mu)]_{\xi_n=i\sigma}\\
&= q^+_{-2}(x',\xi',-i\sigma,\mu) + q^-_{-2}(x',\xi',i\sigma,\mu)\\
&= \frac{1}{(\kappa^++\kappa^-)(\kappa^++\sigma)} + \frac{1}{(\kappa^++\kappa^-)(\kappa^-+\sigma)},\\
\alpha^{(N)}(x',\xi',\mu) &= [h^+q^{\circ N}_{-2N}(x',0,\xi,\mu)]_{\xi_n=-i\sigma} + [h^-q^{\circ N}_{-2N}(x',0,\xi,\mu)]_{\xi_n=i\sigma}\\
&= \tfrac{\partial_\lambda^{N-1}}{(N-1)!}\alpha^{(1)}(x',\xi',\mu);
\end{aligned}
\tag{A.12}
$$

here $\sigma=[\xi']$, and $\alpha^{(N)}$ is weakly polyhomogeneous in $S^{-2N,0,0}\cap S^{0,-2N,0}$ (in fact in $S^{0,0,-2N}$, see below). The crucial information established in [GSc, Lemma 5.5] was that the coefficient of μ^{-2N} in the expansion in powers of μ (as in (3.4)) is 1:

$$
\alpha^{(N)}(x',\xi',\mu) = \mu^{-2N} + \alpha^{(N)}_1(x',\xi',\mu) \text{ with } \alpha^{(N)}_1(x',\xi',\mu)\in S^{1,-2N-1,0};
\tag{A.13}
$$

the remainder $\alpha^{(N)}_1$ is also in $S^{0,-2N,0}$ since $\alpha^{(N)}$ and μ^{-2N} are so.

Lemma A.3. *The symbols $(\kappa^\pm+\sigma)^{-1}$ are weakly polyhomogeneous, belong to $S^{0,0,-1}$ and have N'th λ-derivatives in $S^{0,0,-1-2N}$. For each N, $\alpha^{(N)}$ is weakly polyhomogeneous lying in $S^{0,0,-2N}$.*

Proof. First note that $\kappa^\pm$ and $(\kappa^\pm)^{-1}$ are strongly polyhomogeneous in (ξ',μ) of degree 1 resp. -1, so they lie in $S^{0,0,1}$ resp. $S^{0,0,-1}$. As noted above, they are strongly $(1,2)$-homogeneous in (ξ',λ) of degree 1 resp. -1; hence the N'th λ-derivatives lie in $S^{0,0,1-2N}$ resp. $S^{0,0,-1-2N}$. Similarly, since κ^+ and κ^- lie in a proper subsector of $\{z\in\mathbb{C}\mid \operatorname{Re} z>0\}$, $|\kappa^++\kappa^-|\ge \operatorname{Re}(\kappa^++\kappa^-)\ge c_0|(\xi',\mu)|$ with $c_0>0$, so also $(\kappa^++\kappa^-)^{-1}$ belongs to $S^{0,0,-1}$, with N'th λ-derivatives in $S^{0,0,-1-2N}$.

It was observed in [GSc] that $(\kappa^\pm+\sigma)^{-1}\in S^{-1,0}\cap S^{0,-1}$; we now recall the proof showing how it also assures that $(\kappa^\pm+\sigma)^{-1}\in S^{0,0,-1}$:

Let κ stand for κ^+ or κ^-. Write $(\kappa+\sigma)^{-1}=(\kappa)^{-1}(1+\sigma/\kappa)^{-1}$. We know already that $(\kappa)^{-1}\in S^{0,0,-1}$, so it remains to show that $(1+\sigma/\kappa)^{-1}$ is in $S^{0,0,0}$. Since $\sigma\in S^{1,0,0}$ and $(\kappa)^{-1}\in S^{0,0,-1}\subset S^{-1,0,0}$, σ/κ is in $S^{0,0,0}$, and so is $1+\sigma/\kappa$. Moreover, it is bounded in (ξ',μ) (for $\mu\in\Gamma$, $|\mu|\ge 1$). Since $\operatorname{Re}\kappa>0$, $\sigma<\sigma+\operatorname{Re}\kappa\le|\sigma+\kappa|$, so also the inverse $(1+\sigma/\kappa)^{-1}=\kappa/(\kappa+\sigma)=1-\sigma/(\sigma+\kappa)$ is bounded. Then [GS1, Th. 1.23] (just the beginning of the proof) shows that the inverse $(1+\sigma/\kappa)^{-1}$ does belong to $S^{0,0,0}$.

For the λ-derivatives, we observe that $\partial_\lambda(\kappa+\sigma)^{-1}=-(\kappa+\sigma)^{-2}\partial_\lambda\kappa\in S^{0,0,-3}$ by the composition rules, and hence by successive application, $\partial^k_\lambda(\kappa+\sigma)^{-1}\in S^{0,0,-1-2k}$ for all k.

Using these informations, it is now seen from the form of $\alpha^{(1)}$ in (A.12) that it lies in $S^{0,0,-2}$ with k'th λ-derivatives in $S^{0,0,-2-2k}$, so $\alpha^{(N)}\in S^{0,0,-2N}$. $\square$

Besides $G^{(N)}_\lambda$ (cf. (2.11)), we also need to consider the s.g.o.s $G^\pm(Q^N)$, which arise from compositions such as

$$P_+Q^N_{\lambda,+} = (PQ^N_\lambda)_+ - G^+(P)G^-(Q^N_\lambda). \tag{A.14}$$

Their symbols have the following structure:

Lemma A.4. *The operators $G^{(N)}_\lambda$ and $G^\pm(Q^N)$ have symbols of the form*

$$\begin{aligned} &g(x',\xi,\eta_n,\mu) \sim \sum_{J\in\mathbb{N}} g_{-2N-1-J}(x',\xi,\eta_n,\mu), \text{ with} \\ &g_{-2N-1-J}(x',\xi,\eta_n,\mu) = \sum_{\substack{j\ge 1,\, j'\ge 1\\ j+j'\le 2J+N+1}} \frac{s_{J,j,j',N}(x',\xi',\mu)}{(\kappa+i\xi_n)^j(\kappa'-i\eta_n)^{j'}}; \end{aligned} \tag{A.15}$$

here (κ,κ') equals (κ^+,κ^-) for $G^{(N)}_\lambda$, (κ^+,κ^+) for $G^+(Q^N)$, and (κ^-,κ^-) for $G^-(Q^N)$. The numerators $s_{J,j,j',N}(x',\xi',\mu)$ are strongly homogeneous in (ξ',μ) of degree $j+j'-J-2N-1$. They are in fact functions of (x',ξ',λ), $\lambda=-\mu^2$, strongly $(1,2)$-homogeneous in (ξ',λ) of the indicated degrees, such that differentiation with respect to λ gives a strongly $(1,2)$-homogeneous symbol of 2 steps lower degree.

For the remainders $g'_{-2N-1-M} = g - \sum_{0\le J<M} g_{-2N-1-J}$, the sup-norms in ξ_n resp. (ξ_n,η_n) are bounded by $\langle\xi',\mu\rangle$ in powers $-2N-1-M$, decreasing by $|\alpha|$ for differentiations in (ξ,η_n) of order α, and by 2 for each differentiation in λ (no change of the power for differentiations in x').

Proof. The formulas for the symbols of $G^{(N)}_\lambda$ and $G^\pm(Q^N_\lambda)$ come from [GSc, Prop. 2.5 and 2.3]. The remainder estimates hold since the symbols are strongly polyhomogeneous in (ξ,η_n,μ) so that standard estimates hold when $|\mu|$ is considered as an extra cotangent variable (on each ray). The strong (1,2)-polyhomogeneity (ξ,η_n,λ) assures the statements on λ-derivatives. $\square$

Appendix B. Formulas for compositions with Laguerre functions

Recall the formulas for the (Fourier transformed) Laguerre functions we use in expansions of parameter-independent operators:

$$\hat\varphi'_k(\xi_n,\sigma) = \frac{(\sigma-i\xi_n)^k}{(\sigma+i\xi_n)^{k+1}},\quad \hat\varphi_k(\xi_n,\sigma) = (2\sigma)^{\frac12}\frac{(\sigma-i\xi_n)^k}{(\sigma+i\xi_n)^{k+1}}; \tag{B.1}$$

A priori, σ here might be any positive number, but we will take $\sigma=[\xi']$. The $\hat\varphi_k$ are the *normalized* Laguerre functions, which for $k\in\mathbb{Z}$ form an orthonormal basis of $L_2(\mathbb{R})$. Differentiations in ξ' and ξ_n follow the rules

$$\begin{aligned} \partial_{\xi_j}\hat\varphi_k(\xi_n,\sigma) &= \big(k\hat\varphi_{k-1} - \hat\varphi_k - (k+1)\hat\varphi_{k+1}\big)(2\sigma)^{-1}\partial_{\xi_j}\sigma,\quad j<n,\\ \partial_{\xi_n}\hat\varphi_k(\xi_n,\sigma) &= -i\big(k\hat\varphi_{k-1} + (2k+1)\hat\varphi_k + (k+1)\hat\varphi_{k+1}\big)(2\sigma)^{-1}; \end{aligned} \tag{B.2}$$

here $\partial_{\xi_j}\sigma = \xi_j\sigma^{-1}$ for $|\xi'|\ge 1$.

We have from [GSc] the following formulas for ξ_n-compositions of Laguerre funtions and rational functions involving $\kappa^\pm$:

Lemma B.1. *For all $m \ge 0$ and $j \ge 1$:*
(B.3)
$$\int \frac{(\sigma \pm i\xi_n)^m}{(\sigma \mp i\xi_n)^{m+1}} \frac{1}{(\kappa^\pm(x',\xi',\mu) \pm i\xi_n)^j}\, đ\xi_n = \sum_{m'\ge 0,\, |m'-m|<j} (\kappa^\pm)^{1-j} a^\pm_{jm'} \frac{(\sigma-\kappa^\pm)^{m'}}{(\sigma+\kappa^\pm)^{m'+1}},$$

with universal constants $a^\pm_{jm'}$ that are $O(m^j)$ for fixed j. The resulting expressions are weakly polyhomogeneous ψdo symbols belonging to $S^{0,0,-j}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma)$, with ∂_λ^N of the symbols lying in $S^{0,0,-j-2N}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma)$.

Proof. The formulas are shown in [GSc, Lemma 3.2]. The statements on symbol classes follow since $\sigma = [\xi'] \in S^1 \subset S^{1,0,0}$, $\kappa^\pm \in S^{0,0,1}$ with N'th λ-derivatives in $S^{0,0,1-2N}$, and $(\sigma+\kappa^\pm)^{-1} \in S^{0,0,-1}$ with N'th λ-derivatives in $S^{0,0,-1-2N}$, by Lemma A.3. For the derivatives of the composed expressions one can use the formulas [GSc, (3.21)] for the derivatives of $(\sigma-\kappa^\pm)^m(\sigma+\kappa^\pm)^{-m}$. $\square$

These formulas enter in calculations of compositions such as $GG^{(N)}_\lambda$ and $GG^\pm(Q^N_\lambda)$, where the symbol of G is Laguerre expanded and the rational structure of the symbols of $G^{(N)}_\lambda$ and $G^\pm(Q^N_\lambda)$ (A.15) is used.

Lemma B.2. *The symbol*

$$s_{j,j',m} = \int \frac{(\sigma - i\xi_n)^m}{(\sigma + i\xi_n)^{m+1}} \frac{1}{(\kappa^+ + i\xi_n)^j(\kappa^- - i\xi_n)^{j'}}\, đ\xi_n$$

satisfies for $m \ge 0$, j and $j' \ge 1$:

$$s_{j,j',m} = \sum_{\substack{|m-m'|\le j''<j' \\ m'\ge 0}} b_{jj'j''m'}(\kappa^-)^{-j''} \frac{(\sigma-\kappa^-)^{m'}}{(\sigma+\kappa^-)^{m'+1}} (\kappa^+ + \kappa^-)^{-j-j'+1+j''}, \tag{B.4}$$

where the $b_{jj'j''m'}$ are universal constants that are $O(m^{j'})$ for fixed j, j'. This is a weakly polyhomogeneous symbol in $S^{0,0,-j-j'}$, with N'th λ-derivatives in $S^{0,0,-j-j'-2N}$, for all N. There is a similar formula for $m \le 0$, with j and j' interchanged, κ^+ and κ^- interchanged.

Proof. The formulas were shown in [GSc, Lemma 4.2], and the symbols are analyzed as in the preceding proof. $\square$

These formulas enter in calculations of compositions such as $P_+G^{(N)}_\lambda$ and $P_+G^\pm(Q^N_\lambda)$, where the symbol of P is Laguerre expanded and the rational structure of the s.g.o. is used.

Composition formulas involving ψdo's on the interior contain $h^+_{\xi_n}$- and $h^-_{\xi_n}$-projections. We recall from [GSc, (3.9), (4.8)] that the projections can be removed in certain integrals, e.g.: When $\tilde q$ is a rational function of ξ_n of the form $r(x',\xi',\mu)(\kappa^\pm \pm i\xi_n)^{-j}$, then

$$\begin{aligned} \int \bar{\hat\varphi}_k(\xi_n,\sigma) h^+_{\xi_n}[\tilde q(x',\xi,\mu)\hat\varphi_l(\xi_n,\sigma)]\, đ\xi_n &= \int \bar{\hat\varphi}_k(\xi_n,\sigma)\tilde q(x',\xi,\mu)\hat\varphi_l(\xi_n,\sigma)\, đ\xi_n \\ &= \int h^-_{\xi_n}[\bar{\hat\varphi}_k(\xi_n,\sigma)\tilde q(x',\xi,\mu)]\hat\varphi_l(\xi_n,\sigma)\, đ\xi_n. \end{aligned} \tag{B.5}$$

To see this, note that the integrand in all three expressions is $O(\xi_n^{-2})$ for $|\xi_n| \to \infty$ in $\mathbb{C}$ and meromorphic with no real poles, so that the integral can be replaced by the integral over a large contour in $\mathbb{C}_+$ (the "plus-integral") or the integral over a large contour in $\mathbb{C}_-$. The first equality holds since $\bar{\hat{\varphi}}_k h^-[\tilde{q}\hat{\varphi}_l]$ is meromorphic with no poles in $\mathbb{C}_+$ and is $O(\xi_n^{-2})$ for $|\xi_n| \to \infty$ in $\mathbb{C}$, hence contributes with 0. The second equality holds since $h^+[\bar{\hat{\varphi}}_k\tilde{q}]\hat{\varphi}_l$ is meromorphic with no poles in $\mathbb{C}_-$, and is $O(\xi_n^{-2})$ for $|\xi_n| \to \infty$ in $\mathbb{C}$, hence contributes with 0.

The above formulas were used in [GSc, Sect. 3 and 4] to show that the trace expansions of terms where G or P_+ is composed with one of the parameter-dependent singular Green operators *do not* contribute to the residue coefficient $\tilde{c}_0'$ in (1.10). It was in fact shown that they give expansions where the summation over k as in (1.10) starts with $k \geq 1$, so they do not contribute to the nonlocal coefficient $\tilde{c}_0''$ either. This is important for our present study and will therefore be formulated explicitly:

Proposition B.3. *When G is of class 0 and order $\nu \in \mathbb{R}$, or P is of order $\nu \in \mathbb{Z}$, the operators $GG_\lambda^{(N)}$, $GG^\pm(Q_\lambda^N)$, $P_+G_\lambda^{(N)}$ and $P_+G^\pm(Q_\lambda^N)$ have trace expansions of the form*

$$\sum_{j\geq 1} a_j(-\lambda)^{\frac{n+\nu-j}{2}-N} + \sum_{k\geq 1}(a_k' \log(-\lambda) + a_k'')(-\lambda)^{-\frac{k}{2}-N}. \tag{B.6}$$

Proof. For the compositions with G in front, this is essentially the content of [GSc, Sect. 3], see in particular (3.20) there. We have replaced the index l there by $j = l+1$ in the first sum, $k = l+1$ in the second sum, to facilitate the comparison with (1.10); recall also that $\mu = (-\lambda)^{\frac{1}{2}}$. The arguments there extend immediately to noninteger ν.

For the compositions with P_+ in front, the statement is covered by the calculations in [GSc, Sect. 4], see in particular (4.3) there (which contains neither nonlocal nor logarithmic terms, since $p'_{(l)}$ is a differential operator) and (4.14) there.

In each case, the result is found by showing that in local coordinates, the symbol of tr_n of the operator contains so many negative powers of $\kappa^\pm$ and $\kappa^\pm + \sigma$ that it is in $S^{\nu+1,-2N-1,0} \cap S^{\nu-2N,0,0}$, so that $d = -2N-1$ in (3.6). $\square$

When $\nu \notin \mathbb{Z}$, the log-coefficients a_k' vanish in (B.6), since the degrees of the homogeneous symbols are noninteger.

The conclusions of the proposition hold also if the symbol q of Q_λ is replaced by one of its (x,ξ)-derivatives, since they have a similar structure.

The next lemma is used in calculations of compositions of the type $GQ_{\lambda,+}^N$.

Lemma B.4. *Let*

$$s_{j,l,m}^\pm(y',\xi',\mu) = \int \frac{(\sigma - i\xi_n)^l}{(\sigma + i\xi_n)^{l+1}} \frac{(\sigma + i\xi_n)^m}{(\sigma - i\xi_n)^{m+1}} \frac{1}{(\kappa^\pm \pm i\xi_n)^j}\, đ\xi_n. \tag{B.7}$$

One has for $l, m \in \mathbb{Z}$, $j \geq 1$:

$$\begin{aligned} &\text{For } m < l,\ s_{j,l,m}^+ = 0, \quad s_{j,l,m}^- = \sum_{|l-m-1-m'|<j,\, m'\geq 0} (\kappa^-)^{1-j} b'_{jm'} \frac{(\sigma - \kappa^-)^{m'}}{(\sigma + \kappa^-)^{m'+2}}, \\ &\text{For } m = l,\ s_{j,l,l}^+ = \frac{1}{(\kappa^+ + \sigma)^j 2\sigma}, \quad s_{j,l,l}^- = \frac{1}{(\kappa^- + \sigma)^j 2\sigma}, \\ &\text{For } m > l,\ s_{j,l,m}^+ = \sum_{|m-l-1-m'|<j,\, m'\geq 0} (\kappa^+)^{1-j} b_{jm'} \frac{(\sigma - \kappa^+)^{m'}}{(\sigma + \kappa^+)^{m'+2}}, \quad s_{j,l,m}^- = 0, \end{aligned} \tag{B.8}$$

with $b_{jm'}$ and $b'_{jm'}$ being $O(l^j m^j)$ for fixed j.

When $m \neq l$, the resulting symbols are in $S^{0,0,-j-1}$ having N'th λ-derivatives in $S^{0,0,-j-1-2N}$; for $m = l$, they are in $S^{-1,0,-j}$, with N'th λ-derivatives in $S^{1,0,-j-2N}$.

It follows (cf. (A.12)*) that*

$$\operatorname{tr}_n(\hat{\varphi}_l(\xi_n,\sigma)h^+(q^{\circ N}_{-2N}(x',\xi',\eta_n,\mu)\bar{\hat{\varphi}}_m(\eta_n,\sigma))) = \begin{cases} \alpha^{(N)}(x',\xi',\mu) & \text{if } l = m, \\ s_{lm}(x',\xi',\mu) & \text{if } l \neq m, \end{cases} \tag{B.9}$$

where $s_{lm} \in S^{1,0,-2N-1}$, the symbol seminorms being polynomially bounded in l and m.

Proof. The formulas in (B.8) were shown in [GSc, Lemma 5.2], and the symbols are analyzed as in Lemma B.1.

Formula (B.9) is deduced from (B.7)–(B.8) using (B.5) and noting that (B.7) contains non-normalized Laguerre functions $\hat{\varphi}'_l$ and $\bar{\hat{\varphi}}'_m$ so that we get an extra factor 2σ from the normalized Laguerre functions. The statements for $l \neq m$ follow straightforwardly from the descriptions in (B.8). The statements for $l = m$ were proved in [GSc, Prop. 5.4]; they can be verified very simply by doing the calculation for $N = 1$ and passing via λ-derivatives to the general case. ☐

The important point in this lemma is that a symbol contributing to $\tilde{c}'_0$ and $\tilde{c}''_0$ does appear, but with a special form that allows further clarification (using the information (A.13)).

We summarize some results from [GSc] on compositions of the form $GQ^N_{\lambda,+}$ in the following statement:

Proposition B.5. *Consider $GQ^N_{\lambda,+}$ in a localization to $\mathbb{R}^n_+$. Here*

$$\operatorname{tr}_n(GQ^N_{\lambda,+}) = \widetilde{S}_0 + \widetilde{S}_1 \text{ with } \widetilde{S}_0 = \operatorname{OP}'(\operatorname{tr}_n g(x',\xi')\alpha^{(N)}(x',\xi',\mu)), \tag{B.10}$$

where $\widetilde{S}_0$ and $\widetilde{S}_1$ are ψdo's on $\mathbb{R}^{n-1}$ with symbols in $S^{\nu,-2N,0} \cap S^{\nu-2N,0,0}$ resp. $S^{\nu+1,-2N-1,0} \cap S^{\nu-2N,0,0}$. The traces $\operatorname{Tr}_{\mathbb{R}^{n-1}}(\widetilde{S}_0)$ and $\operatorname{Tr}_{\mathbb{R}^{n-1}}(\widetilde{S}_1)$ have expansions

$$\sum_{j\geq 1} a_j(-\lambda)^{\frac{n+\nu-j}{2}-N} + \sum_{k\geq 0}(a'_k \log(-\lambda) + a''_k)(-\lambda)^{-\frac{k}{2}-N}, \tag{B.11}$$

where the sum over k starts at $k = 1$ for $\widetilde{S}_1$, and the value of a'_0 can be determined more precisely for $\widetilde{S}_0$.

Proof. This is proved in [GSc, Sect. 5], see in particular Prop. 5.3, 5.4, and Sect. 5.b there. ☐

We have in fact in view of Lemma A.3 that $\widetilde{S}_0 \in S^{\nu,0,-2N}$, and also the information on $\widetilde{S}_1$ can be upgraded to $S^{\nu+1,0,-2N-1}$ by a closer analysis (as in Section 3 in this paper, using methods from Proposition 3.8 to handle the case where q depends on x_n).

Let us introduce a notation for the Poisson and trace operators on $\mathbb{R}^n_+$ (Laguerre operators) defined from Laguerre functions:

$$\Phi_j = \operatorname{OPK}(\hat{\varphi}_j(\xi_n,\sigma)), \quad \Phi^*_k = \operatorname{OPT}(\bar{\hat{\varphi}}_k(\xi_n,\sigma)). \tag{B.12}$$

Here Φ_j maps $L_2(\mathbb{R}^{n-1})$ continuously (in fact isometrically) into $L_2(\mathbb{R}^n_+)$, and its adjoint is $\operatorname{OPT}(\bar{\hat{\varphi}}_j)$. Moreover, because of the orthonormality of the $\hat{\varphi}_j$,

$$\Phi^*_j\Phi_k = \delta_{jk}I, \tag{B.13}$$

where I is the identity operator on functions on $\mathbb{R}^{n-1}$.

Lemma B.6. *A singular Green operator G on $\mathbb{R}^n_+$ of order ν and class 0 can be written in the form*

$$G = \sum_{j,k\in\mathbb{N}} \Phi_j C_{jk} \Phi_k^*, \quad C_{jk} = \mathrm{OP}'(c_{jk}(x',\xi')), \tag{B.14}$$

with a rapidly decreasing sequence $(c_{jk})_{j,k\in\mathbb{N}}$ in S^ν.

Proof. The symbol g of G has a Laguerre expansion:

$$g(x',\xi',\xi_n,\eta_n) = \sum_{j,k\in\mathbb{N}} d_{jk}(x',\xi')\hat{\varphi}_j(\xi_n,\sigma)\bar{\hat{\varphi}}_k(\eta_n,\sigma), \tag{B.15}$$

with $(d_{jk})_{j,k\in\mathbb{N}}$ rapidly decreasing in S^ν, i.e., the relevant symbol seminorms on $\langle k\rangle^N\langle j\rangle^{N'} d_{jk}$ are bounded in j,k for any $N,N'\in\mathbb{N}$. Then

$$G = \sum_{k\in\mathbb{N}} \mathrm{OPK}\,\Big(\sum_{j\in\mathbb{N}} d_{jk}(x',\xi')\hat{\varphi}_j(\xi_n,\sigma)\Big) \circ \mathrm{OPT}\,\big(\bar{\hat{\varphi}}_k(\eta_n,\sigma)\big).$$

Each Poisson operator $\mathrm{OPK}(\sum_{j\in\mathbb{N}} d_{jk}(x',\xi')\hat{\varphi}_j(\xi_n,\sigma))$ has a symbol in y'-form, $l_k(y',\xi)$. Since the d_{jk} are rapidly decreasing in S^ν, the sequence (l_k) is rapidly decreasing in the topology of Poisson symbols of order $\nu+\frac{1}{2}$. We now expand l_k in a Laguerre series:

$$l_k(y',\xi) = \sum_{j\in\mathbb{N}} c_{jk}(y',\xi')\hat{\varphi}_j(\xi_n,\sigma),$$

and conclude from [G2, Lemma 2.2.1] that (c_{jk}) is rapidly decreasing in S^ν. Since

$$\mathrm{OPK}(l_k) = \sum_{j\in\mathbb{N}} \mathrm{OPK}(\hat{\varphi}_j(\xi_n,\sigma))\,\mathrm{OP}'(c_{jk}(x',\xi')),$$

we obtain

$$G = \sum_{j,k\in\mathbb{N}} \mathrm{OPK}\,\big(\hat{\varphi}_j(\xi_n,\sigma)\big)\,\mathrm{OP}'\,\big(c_{jk}(x',\xi')\big)\,\mathrm{OPT}\,\big(\bar{\hat{\varphi}}_k(\eta_n,\sigma)\big),$$

which shows the assertion. $\square$

The last part of the proof shows the following useful result:

Lemma B.7. *A Poisson operator K of order $\nu+\frac{1}{2}$ on $\overline{\mathbb{R}}^n_+$ can be written $K=\sum_{j\geq 0}\Phi_j C_j$ with a rapidly decreasing sequence (C_j) of ψdo's with symbols in S^ν.*

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